INTEGRATION

BY

TRIGONOMETRIC AND IMAGINARY SUBSTITUTION

BY

CHARLES O. GUNTHER, M.E.

Assistant Professor of Mathematics and Mechanics in the Stevens Institute of Technology

WITH AN INTRODUCTION

BY

J. BURKITT WEBB, C.E.

Professor of Mathematics and Mechanics in the Stevens Institute of Technology

NEW YORK

D. VAN NOSTRAND COMPANY

23 MURRAY AND 27 WARREN STREETS

1907

PREFACE.

A METHOD of integration, which may be called the "*Triangle Method*," has been used successfully for the past few years by the author in his classes. The primary object in view has been to eliminate the "Reduction Formulæ," and make the student independent of text-books and tables of integrals. This method is founded upon trigonometric principles with the result that the student gains proficiency not only in the integration of trigonometric differential expressions but also in the transformation of algebraic expressions into trigonometric and exponential ones, and *vice versa*.

This book is intended to be used in conjunction with the usual text-books on the subject, and should be taken up after the student has become familiar with the simple rules of integration, resulting from the reversion of the rules for differentiation.

CHAS. O. GUNTHER.

GRAND VIEW-ON-HUDSON, N. Y.,
January 1, 1907.

CONTENTS.

INTRODUCTION.

	PAGE.
Introduction	1
Imaginary Quantities	2
Exponentials	9
Explanation of Figures and Tables	12
Analytical Trigonometry	14
Interrelations	16
Integration	18
Classification	21

CHAPTER I.

Trigonometric Differentials.

Arts.		
1.	Trigonometric Formulae	27
2.	Trigonometric Differentials	29
3–8.	Integration of $dy = \cos^m x \sin^n x dx$	30
3.	When either m or n is an odd positive integer	30
4.	When $m+n$ is an even negative integer	31
5.	When both m and n are even positive integers	32
7.	When $m+n$ is an odd negative integer	34
8.	When either m or n is an even positive integer, the other being a negative integer, and $m+n$ is a positive integer or zero	39
9.	Integration of $dy = \cos mx \cos nx\, dx$, $dy = \sin mx \cos nx\, dx$, and $dy = \sin mx \sin nx\, dx$	41
10–12.	Integration of $dy = \epsilon^{ax} \cos bx\, dx$, $dy = \epsilon^{ax} \sin bx\, dx$	42
	Miscellaneous Examples I	46

CHAPTER II.

Rationalization by Trigonometric Substitution.

Arts.		Page.
13.	Introduction	47
14.	Rationalization of Expressions containing $\sqrt{a^2+x^2}$	47
15.	" " " " $\sqrt{a^2-x^2}$	49
16.	" " " " $\sqrt{x^2-a^2}$	50
17.	Change of Form of Radical	52
	Miscellaneous Examples II.	58
18.	Rationalization of Expressions Containing $\sqrt{2ax-x^2}$	59
19.	" " " " $\sqrt{2ax+x^2}$	60
20.	" " " " trinomial surds	62
21.	Binomial Differentials	63
	Answers to Examples	65

INTRODUCTION.

Some time since I suggested to Professor Gunther that the instruction in Integral Calculus might be improved by teaching the student to integrate for himself a variety of expressions by the use of imaginary and trigonometric substitutions, integration by parts and other methods, such as I use myself in the higher classes and which can easily be remembered, and giving up the use of tables of integrals and the customary formulæ of reduction, especially the four standard formulæ for raising and lowering the exponent, which are too cumbersome and difficult to remember and use. This will go far toward making the student independent of text-books in his integrations.

Professor Gunther has shown such appreciation of these methods as to work them out with great care in a series of problems covering the ground of the ordinary text-book, leaving it to me to give a more general view of the principles involved as an introduction thereto.

This introduction is therefore intended more for the teacher than the student, but it is placed here so that it may be in the hands of every student and thus reach those rare and thoughtful ones found in most classes who aim to master all the principles involved in a subject as well as the practical part.

The most elementary acquaintance with as much of higher mathematics as is becoming necessary in the arts, requires

familiarity with imaginary and complex * quantities, which ought now to be included in elementary algebra, as well as with exponentials and with analytical trigonometry in connection with exponential quantities. Students come to the higher classes with much less practice in trigonometry and exponentials than they have had in algebra, and with none in imaginary quantities, and the practice of integrating all algebraic forms algebraically, and even transforming some of the trigonometric forms into algebraic, only increases the defect. It is therefore a great advantage to reverse the procedure and give the student practice in analytic trigonometry, and other fundamentals of higher mathematics, in the integration of algebraic forms by trigonometrical and other suitable methods.

IMAGINARY QUANTITIES.

Whether the nature and use of $i = \sqrt{-1}$ should be made familiar to the student of ordinary algebra is scarcely a question, because there is no possibility of leaving it out; and the only question is; Shall he be made familiar with it as an impossible quantity, when its existence shows it to be possible, and shall it be called an imaginary quantity and no attempt be made to assist his imagination to grasp it, or shall explanations be given showing it to be a reasonable and useful quantity? If in an exact science like algebra the logical development of the fundamental assumptions can lead to anything impossible, grave doubt is cast upon their correctness or the accuracy of the reasoning employed.

* A complex quantity is the sum of a real quantity and an imaginary, *i.e.*, it is a quantity part real and part imaginary.

The explanation required is not difficult. The square root of any minus number can be factored into the product of the square root of the number itself and $\sqrt{-1}$, thus: $\sqrt{-4} = 2\sqrt{-1}$, so that the only thing requiring explanation is $\sqrt{-1}$. First then, if -1 be recognized as a quantity, $\sqrt{-1}$ must also be a quantity, for it is inconceivable that the quantitative nature of -1 can be destroyed by the operation of finding its square root when this operation has no such effect on other quantities; or, if conceivable, -1 must be a quantity of a different nature from others, the difference being responsible for this peculiar action of the square root upon it.

What is, then, the nature of a square root, and what the difference between $+1$ and -1?

The square root of a quantity is one of the two equal factors whose product is the quantity, thus:

If $\quad b \times b = a$, b is the square root of a, consequently:

If the square root of a quantity be used twice as a factor it produces the same result as using the quantity once as a factor,

thus: $$c \times b \times b = c \times a.$$

To appreciate clearly the difference between $+1$ and -1 it must be remembered that in algebra plus and minus are used in two ways, so that it is sometimes advantageous to distinguish them from each other. In the expressions $m + n$ and $m - n$ the signs are supposed to represent addition and subtraction, and this is the first idea of them when they are used in arithmetic; so that while $5 - 3$ has a meaning, $3 - 5$ is held to be meaningless, for from a pile of 5 balls 3 can be taken away, but from a pile of 3 it is impossible to take 5. In algebra, however, a broader view is taken.

In arithmetic four fundamental operations are taught, with a process for performing each, — addition, subtraction, multi-

plication, and division; but in algebra these are reduced to two only, thus: if $m=5$ and $n=3$, arithmetic teaches how to

add n to m,	$m+n=5+3=8$	(a)
subtract n from m,	$m-n=5-3=2$	(b)
multiply m by n,	$m\times n=5\times 3=15$ . . .	(c)
divide m by n,	$m\div n=5\div 3=1\frac{2}{3}$ . . .	(d)

But in algebra, if n is to be subtracted from m, no separate operation of subtraction is recognized, the rule being to *change the sign of n and proceed as in addition;* and in the case of n added to m, symmetry of statement leads to, *leave the sign unchanged and proceed as in addition,* as the corresponding rule.

There must, then, be a sign belonging to quantities, which can be changed or left unchanged, and this sign is called the *intrinsic sign*, which may be + or − ; besides which a second or *copulative sign* is needed to express the addition, and this is always plus. The algebraic expression of (a) will therefore be

$$^{+}m+{}^{+}n=m+n=8 \quad . \quad . \quad . \quad . \quad . \quad (a')$$

and of (b)

$$^{+}m+{}^{-}n=m-n=2 \quad . \quad . \quad . \quad . \quad . \quad (b')$$

where $m+n$ and $m-n$ may be regarded as abbreviated forms which generally lead to no ambiguity.

In the same way algebra dispenses with division, for to divide m by n it is sufficient to write n as a fraction and multiply, thus:

$$m\times\frac{1}{n}=\frac{m}{n}=1\tfrac{2}{3}.$$

A much closer parallel appears, however, in the customary algebraic use of exponents. An exponent is a small numeral at the upper right-hand corner of a quantity which states two

things, first, that the quantity or number is to be used as a factor, and second, the number of times it is to be so used. According to this usage (c) may be written algebraically

$$m^1 \times n^1 \ = m \,.\, n \ = mn \quad = 15 \qquad \ldots \quad (c')$$

and (d) $$m^1 \times n^{-1} = m \,.\, n^{-1} = mn^{-1} = 1\tfrac{2}{3} \qquad \ldots \quad (d')$$

for the rule is to *change the sign of the exponent of* n *and proceed as in multiplication,* whereas in (c') we *leave the sign unchanged and proceed as in multiplication.* The forms $m \,.\, n$, mn, etc., may be regarded as abbreviations, which generally lead to no ambiguity. Here instead of the copulative sign of addition there is that of multiplication, $\times$, or the dot used in algebra, and instead of the intrinsic signs of the quantities there are the signs of the exponents.

The failure to keep in mind and distinguish between the copulative and intrinsic signs in addition is a source of confusion in many problems, as, for instance, in logarithms when the intrinsic sign of the characteristic is different from that of the mantissa, and it would be advantageous if separate symbols were provided for them.

It would be better if the copulative sign for addition were $-$, so that $m - n$ should mean m plus n. It is the simpler mark of the two, and $m - n$ could be changed to $m + n$ by adding a stroke; whereas with the signs as they now are, the change from m plus n to m minus n requires an erasure, and this change has to be made much oftener than the contrary one.

It would also be better in algebraic multiplication, which includes division, to use the line already employed to indicate a fraction, or, in other words, the line employed to indicate division, for the general sign of multiplication, thus ; $\frac{m}{n}$ means

m divided by n, or m multiplied by $\frac{1}{n}$. Why not write $m\underline{n}$ or $\underline{mn}$ for m multiplied by n, so that while a line in front of a quantity would indicate that it was to be added, as in $m-n$ (n added to m), a line under it would mean that it was a factor? The algebraic rule dispensing with division and reducing it to multiplication would then be similar to that for subtraction and would read: To divide by a quantity, *change the factoral sign from under to over it and proceed as in multiplication,* thus: $\underline{m}$ and $\underline{n}$ are both factors, and $\underline{mn}$ are two factors multiplied together; but if m is to be a multiplier and n a divisor, then they must be written $\underline{m}$ and $\overline{n}$, and when multiplied together would appear as $\underline{m}\overline{n}$ or $\frac{m}{n}$ or m/n. (c) would then be written thus:

$$\underline{mn} = 5 \times 3 = 15,$$

and (d) thus:

$$\underline{m}\overline{n} = \frac{m}{n} = 5 \div 3 = \frac{5}{3} = 1\tfrac{2}{3}.$$

Now as to the difference between $+1$ and -1, these signs are evidently *intrinsic signs,* and the -1 is not to be thought of as 1 subtracted from anything. What, then, do these intrinsic signs indicate? It will be found upon consideration of this question that the intrinsic sign of some physical quantities is by nature plus and cannot be minus. A cubic foot of water has about two units of mass in it, and the same volume of a substance of half that density would have one unit, and, further, in a cubic foot of vacuum there would be zero units of mass; but to have -1 or -2 units of mass in a cubic foot is inconceivable, and may well be said to be impossible, so that the *intrinsic sign of density is plus.* Also the intrinsic sign of the radius of a sphere or a circle is $+$, and

the device of a minus value to the radius is only such, and sometimes leads to error. There are, however, quantities to which nature allows both plus and minus intrinsic signs.

In rectangular co-ordinates the intrinsic sign of either x or y may be plus or minus, $+$ meaning a distance measured in an assumed direction, say to the right of the origin, and $-$ meaning in the opposite or left-hand direction, and in this way a rational interpretation of $+1$ and -1 is obtained as *two unit distances measured in opposite directions from an origin or zero point.* They may therefore be properly represented by arrows of unit length drawn to the right and left, thus:

$$-1 \longleftarrow\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!- \circ -\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!\longrightarrow +1$$

an arrow (or directed line, or vector) having two properties and no more, magnitude or length, and direction, the same as quantities $+1$ and -1 have.

$+1$ and -1 may then be regarded as referring not only to the unit distances themselves, but to the points at the ends of the arrows.

Now -1 or $+1$ regarded as a factor must be somewhat different from the -1 and $+1$ regarded as points or distances from the origin. Indeed, in the ordinary multiplication of 5 by 3, 5 is termed the multiplicand and 3 the multiplier, and 3 is supposed to operate as a factor on 5, 5 being trebled by the operation and not regarded as a factor, so that a difference is recognized, which can be further emphasized by considering that there would be no difficulty in multiplying -1 by 3, but only in the reverse operation. To get out of the difficulty it might have to be assumed that the reversal should make no difference in the result, but this would not help the matter if -1 were to be multiplied by -3.

But in algebra factors can be minus as well as plus, and a remarkable rule is given for their multiplication. The num-

bers or quantities themselves are to be multiplied in the usual way for any number of factors, as $^+m \cdot {}^-n \cdot {}^-p \cdot {}^-q$, and *a sign is to be prefixed to the product,* + *if the number of minus signs is even, and* – *if it is odd*, and this is an arbitrary rule for which no justification is given except what it gets subsequently from the fact that it works well.

Now this rule of sign might equally well have been arbitrarily stated in either of the following forms:

Call a plus sign zero and a minus sign one, and add together the signs of the factors; if the result is an even number put +, *and if an odd number put* – *before the product;* or, supposing angular measure to be understood:

Call a plus sign 0° *and a minus sign* 180°, *and add together the signs of the factors; if the result is a multiple of* 360° *the product is plus, and if not it is minus.*

It must then be evident that the rule given is equivalent to a statement that the effect of the factor -1 is to reverse the direction of the quantity on which it operates without altering its magnitude, which can only be done by revolving it about the origin through 180°.

Therefore the use of -1 as a factor does not alter its compound nature; it is still a directed quantity, and as such may properly be represented by a line drawn to the left of the origin; but instead of the minus sign meaning simply the direction to the left, it now represents a revolution of 180°, and its operation as a factor is to revolve another quantity (considered as a multiplicand) through 180°, thus changing it from whatever direction it may have to the opposite one.

The meaning of the square root of minus one and its effect follow directly from the foregoing and the definition of square root. The quantity $\sqrt{-1}$ used twice as a factor must be equivalent to -1 used once, and cause a revolution of 180°; it must therefore, when used once, cause a revolution of 90°.

Starting then with the directed line or arrow $+1$ as the multiplicand,

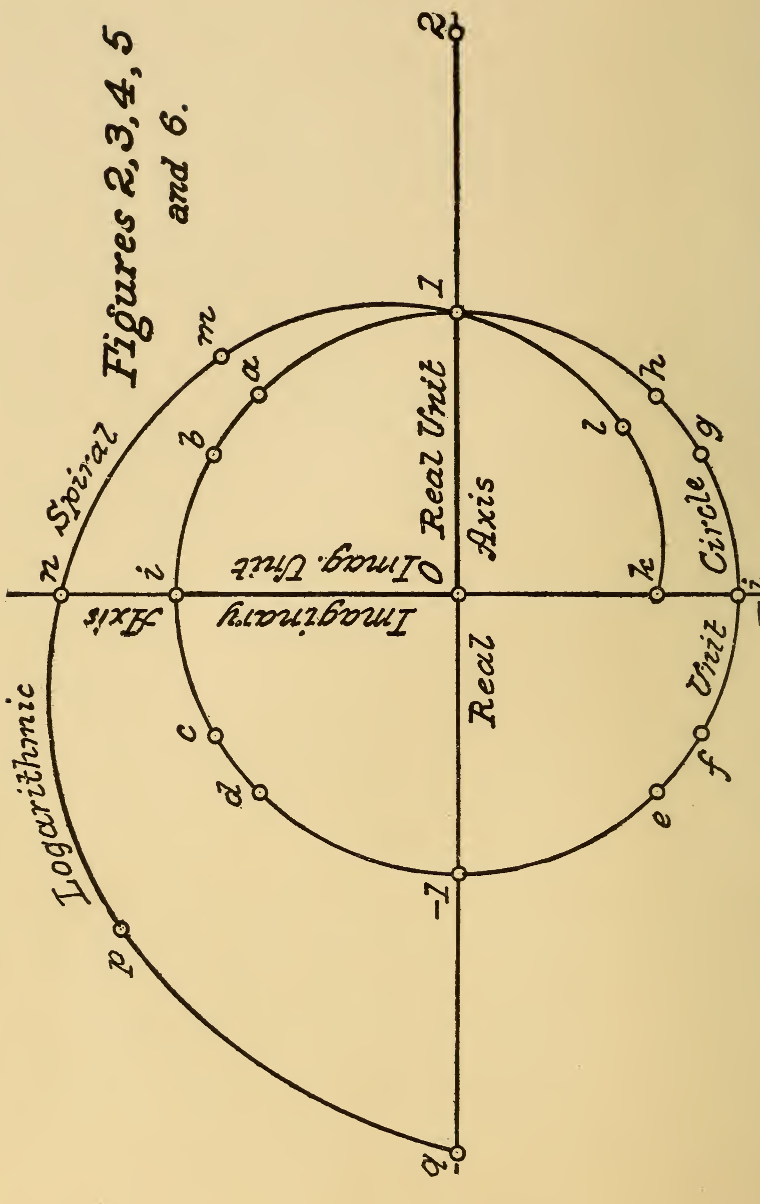
$$1 \cdot \sqrt{-1} = \sqrt{-1},$$
$$\sqrt{-1} \cdot \sqrt{-1} = -1,$$
$$-1 \cdot \sqrt{-1} = -\sqrt{-1},$$
and
$$-\sqrt{-1} \cdot \sqrt{-1} = 1.$$

As a directed quantity $\sqrt{-1}$ is therefore unity measured in the direction of the Y-axis, and as a factor it is a revolution through a right angle in that angular direction which is considered positive. i is used to represent the plus square root of minus one, or $i = \sqrt{-1}$, and is called the *imaginary unit* to distinguish it from $+1$ which is called the *real unit*, although one is just as real as the other, differing in direction only, as do the co-ordinates x and y. $i = \sqrt{-1}$ is therefore an actual algebraic quantity as easy to use after a little practice as any other, and very useful in many cases. It furnishes an elegant algebraic method of working with x and y co-ordinates, by means of *complex quantities*, without having to keep them separate from each other, and includes also polar co-ordinates by the help of exponentials.

All points in a plane have therefore their algebraic representation as shown in the following figures. See page 10.

EXPONENTIALS.

Familiarity with exponential quantities is important in mathematical work of all kinds. The base ε of the *Natural System of Logarithms* was not invented, any more than $\sqrt{-1}$ was, to plague the brains of unwilling students, but it is a *Constant of Nature*, the same as π is, and was discovered, as any rare mineral might be unearthed, by mathematical digging. Like a rare mineral, a constant of nature



Figures 2, 3, 4, 5 and 6.

Points.	Fig. 2.	Fig. 3.	Fig. 4.	Fig. 5.	Fig. 6, $\mathcal{E}^x$
0	$x^{-\infty}$	$i^{-\infty}$	$0 \quad +i0$	$0(7x\pi+i\,\Lambda x\pi)$	$x=-\infty$
1	1^x or x^0	i^0	$1 \quad +i0$	$70\pi+i\,\Lambda 0\pi$	$=\ 0$
a	$1^{1/8}$, $(-1)^{1/4}$	$i^{1/2}$	$2^{-1/2}+i2^{-1/2}$	$7\tfrac{1}{4}\pi+i\,\Lambda\tfrac{1}{4}\pi$	$=\ i\tfrac{1}{4}\pi$
b	$1^{1/6}$, $(-1)^{1/3}$	$i^{2/3}$	$2^{-1}+i(3/4)^{1/2}$	$7\tfrac{1}{3}\pi+i\,\Lambda\tfrac{1}{3}\pi$	$=\ i\tfrac{1}{3}\pi$
i	$1^{1/4}$, $(-1)^{1/2}$	i	$0 \quad +i$	$7\tfrac{1}{2}\pi+i\,\Lambda\tfrac{1}{2}\pi$	$=\ i\tfrac{1}{2}\pi$
c	$1^{1/3}$, $(-1)^{2/3}$	$i^{4/3}$	$-2^{-1}+i(3/4)^{1/2}$	$7\tfrac{2}{3}\pi+i\,\Lambda\tfrac{2}{3}\pi$	$=\ i\tfrac{2}{3}\pi$
d	$1^{3/8}$, $(-1)^{3/4}$	$i^{3/2}$	$-2^{-1/2}+i2^{-1/2}$	$7\tfrac{3}{4}\pi+i\,\Lambda\tfrac{3}{4}\pi$	$=\ i\tfrac{3}{4}\pi$
-1	$1^{1/2}$, $(-1)^{4/4}$	i^2	$-1 \quad +i0$	$7\tfrac{4}{4}\pi-i\,\Lambda\tfrac{4}{4}\pi$	$=\ i\tfrac{4}{4}\pi$
e	$1^{5/8}$, $-(-1)^{1/4}$	$-i^{1/2}$	$-2^{-1/2}-i2^{-1/2}$	$7\tfrac{5}{4}\pi+i\,\Lambda\tfrac{5}{4}\pi$	$=\ i\tfrac{5}{4}\pi$
f	$1^{2/3}$, $-(-1)^{1/3}$	$-i^{2/3}$	$-2^{-1}-i(3/4)^{1/2}$	$7\tfrac{4}{3}\pi+i\,\Lambda\tfrac{4}{3}\pi$	$=\ i\tfrac{4}{3}\pi$
-i	$1^{3/4}$, $-(-1)^{1/2}$	$-i$	$0 \quad -i$	$7\tfrac{3}{2}\pi+i\,\Lambda\tfrac{3}{2}\pi$	$=\ i\tfrac{3}{2}\pi$
g	$1^{5/6}$, $-(-1)^{2/3}$	$-i^{4/3}$	$2^{-1}-i(3/4)^{1/2}$	$7\tfrac{5}{3}\pi+i\,\Lambda\tfrac{5}{3}\pi$	$=\ i\tfrac{5}{3}\pi$
h	$1^{7/8}$, $-(-1)^{3/4}$	$-i^{3/2}$	$2^{-1/2}-i2^{-1/2}$	$7\tfrac{7}{4}\pi+i\,\Lambda\tfrac{7}{4}\pi$	$=\ i\tfrac{7}{4}\pi$
k	$(-2)^{-1/2}$	$-i2^{-1/2}$	$0 \quad -i2^{-1/2}$	$.707(7\tfrac{3}{2}\pi+i\,\Lambda\tfrac{3}{2}\pi)$	$=-\tfrac{1}{2}(\log 2+i\pi)$
l	$(-2)^{-1/4}$	$-i^{3/2}2^{-1/4}$	$2^{-3/4}-i2^{-3/4}$	$.840(7\tfrac{7}{4}\pi+i\,\Lambda\tfrac{7}{4}\pi)$	$=-\tfrac{1}{4}(\log 2+i\pi)$
1	$(-2)^0$	$-i^2 2^0$	$1 \quad +i0$	$70\pi+i\,\Lambda 0\pi$	$=\ \tfrac{1}{\infty}(\log 2+i\pi)$
m	$(-2)^{1/4}$	$i^{1/2}2^{1/4}$	$2^{-1/4}+i2^{-1/4}$	$1.189(7\tfrac{1}{4}\pi+i\,\Lambda\tfrac{1}{4}\pi)$	$=\ \tfrac{1}{4}(\log 2+i\pi)$
n	$(-2)^{1/2}$	$i2^{1/2}$	$0 \quad +i2^{1/2}$	$1.414(7\tfrac{1}{2}\pi+i\,\Lambda\tfrac{1}{2}\pi)$	$=\ \tfrac{1}{2}(\log 2+i\pi)$
p	$(-2)^{3/4}$	$i^{3/2}2^{3/4}$	$-2^{1/4}+i2^{1/4}$	$1.682(7\tfrac{3}{4}\pi+i\,\Lambda\tfrac{3}{4}\pi)$	$=\ \tfrac{3}{4}(\log 2+i\pi)$
q	-2	$i^2 2$	$-2 \quad +i0$	$2(7\tfrac{4}{4}\pi+i\,\Lambda\tfrac{4}{4}\pi)$	$=\ \tfrac{4}{4}(\log 2+i\pi)$
2	2	$i^0 2$	$2 \quad +i0$	$2(70\pi+i\,\Lambda 0\pi)$	$=\ \log 2$

EXPLANATION OF FIGURES AND TABLE.

Figures 2 to 6 differ only in the values ascribed to the points shown. These values could not easily be written upon the figures, so that instead of drawing five separate figures, each with its set of values, but one is shown with reference letters at the points. These letters are placed in the first column of the Table, headed "Points," and in the following columns the values belonging to the points are given for each of the five figures.

In Figure 2 each point, or the vector from the origin to the point is a power of a real number, thus; the origin zero is $x^{-\infty} = 0$ where x is any real number, and point 1 is unity to any power x or x^0; a, b, etc., are given both as powers of $+1$ and -1 and the points on the spiral as powers of -2. Other values also belong to the points, thus; one of the four fourth roots of -1 is a, the other three are d, e and h. The three cube roots of -1 are b, -1 and g, and the two square roots i and $-i$. The four fourth roots of $+1$ are $1, i, -1$ and $-i$, the three cube roots are 1, c and f, and the two square roots are 1 and -1.

In Figure 3 each point is a power of the Imaginary Unit i or a multiple thereof, so that the points can be represented by powers of either real or imaginary numbers at will.

In Figure 4 each point is regarded as a Complex Quantity z, given in the Standard Algebraic Form $z = x + iy$, x and y being its Cartesian Coördinates, thus; b has $x = 2^{-1}$ and $y = 3^{-\frac{1}{2}}$.

In Figure 5 the Trigonometric Form $z = \rho$ (cosine $x + i$ sine x) is used, thus; n has $\rho = 1.414$ and $x = \frac{1}{2}\pi$. The symbols used in this column stand, as might be inferred, for the words "cosine" and "sine," the use of words or abbreviations thereof in a formula being regarded as inelegant and wasteful of room.

In Figure 6 the Exponential Form $z = \varepsilon^x$ is given. To save room x only is tabulated.

possesses wonderful and valuable properties, which are discovered upon examination and analysis. This constant ε is represented by one of the simplest of series, for, as shown by the Differential Calculus,

$$\varepsilon^x = x^0 + x^1 + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \ . \ . \ . \text{ etc.,}$$

and therefore,

$$\varepsilon = \varepsilon^1 = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \ . \ . \ . \text{ etc.,}$$

each term being divided by its number to form the next term.

A more comprehensive idea of ε^x must be taken than that given in connection with ordinary logarithmic and exponential work. It is there taught that a minus quantity has no logarithm, which may seem true from the fact that the real numbers from $-\infty$ to $+\infty$ are all needed as logarithms for the plus numbers, integral and fractional, between 0 and $+\infty$. Where, then, can logarithms for minus numbers be found?

In the series for ε^x if x be made imaginary, say $x = i\theta$, θ being real, $\varepsilon^x = \varepsilon^{i\theta}$ as will be seen later, will always be a point on the unit circle (or circle of unit radius) having the origin for its center, and in the same way $r\varepsilon^{i\theta}$ is a point on the circle of radius r, also it will be found that

$$\varepsilon^{i\pi} = -1, \qquad \text{or,}$$

$$i\pi = \text{logarithm of } -1; \qquad \text{further,}$$

$$r\varepsilon^{i\pi} = \varepsilon^{\log r + i\pi} = -r, \qquad \text{therefore}$$

$$\log r + i\pi = \log(-r),$$

so that the logarithm of a minus number is the logarithm of the same plus number plus $i\pi$.

There is, then, an important connection between these two constants of nature, ε and π, and other numbers. ε has also other important trigonometrical connections, especially in

connection with i, i being in fact the foundation quantity in a sort of trigonometry in which angles are represented and treated algebraically along with distances, as may be inferred from the last two figures. This makes it necessary to generalize and extend the principles of trigonometry.

ANALYTICAL TRIGONOMETRY.

The two fundamental functions in analytical trigonometry are the cosine and its complementary function the sine, and the Differential Calculus expresses them both as series of powers of the angle x. But the idea of these quantities obtained in ordinary trigonometry, as the ratios between the sides of a triangle, must be generalized for them to be of use in higher mathematics; and to do this we drop the old definitions and define these functions of x anew so as to *generalize* and *extend* their meanings. The cosine and sine are therefore defined as the sums of their series of powers of x, thus:

$$\text{cosine } x = x^0 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \ldots \text{ etc.,}$$

and

$$\text{sine } x = x^1 - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \ldots \text{ etc.}$$

The cosine is written first, contrary to the usual custom, because it is the simpler and more important function, and corresponds with the x direction or *real axis*, while the sine corresponds with the y direction or *imaginary axis.*

These definitions are a generalization of the functions and include the old definitions. In this connection the nature of x must be understood and a false idea of it corrected. x is usually regarded as an angle, but in reality so to regard it is nothing more than to remember that it refers to an angle in

the particular problem concerned, for mathematically x cannot be anything more than the number expressing the angle in radians. The generalization of the cosine and sine therefore consists in calling the sums of the above series by the same names, "cosine of x" and "sine of x," even when x is a number obtained by measuring some quantity other than an angle.

This general view allows us also to extend the meaning of the functions to include imaginary and complex values of x, and, without trying to form any idea as to what an imaginary or complex angle could or might be, it is allowable for convenience to still call x "the angle x" when it is not an angle, and "the imaginary angle x" when it is an imaginary or complex number.

In ordinary trigonometry it is taught that cosines and sines all lie between $+1$ and -1, so that a cosine equal to 2 would be an impossibility, but by supposing x to be imaginary it becomes possible, and cosines and sines can have all values from $+1$ to $+\infty$; also by giving x a suitable complex value these functions take values between -1 and $-\infty$, other complex values for x giving complex values for the functions, so that any point in the co-ordinate plane, real, imaginary, or complex, can be represented in analytical trigonometry by the cosine or sine of some x, but this x can only represent a real angle in the case that the functions lie between $+1$ and -1.

The effect of an imaginary value of x on the cosine and sine and on ε^x will appear by placing $x = i\theta$ in the series and simplifying the results; thus it will be found that

$$\text{cosine } x = \cos i\theta = 1 + \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}} + \frac{\theta^5}{\underline{|5}} + \ldots \text{ etc.,} \qquad \text{and}$$

$$\text{sine } x = \sin i\theta = i\left(\theta + \frac{\theta^3}{\underline{|3}} + \frac{\theta^5}{\underline{|5}} + \frac{\theta^7}{\underline{|7}} + \ldots \text{ etc.}\right),$$

to which may be added

$$\varepsilon^x = \varepsilon^{i\theta} = 1 + i\theta - \frac{\theta^2}{\lfloor 2} - i\frac{\theta^3}{\lfloor 3} + \frac{\theta^4}{\lfloor 4} + i\frac{\theta^5}{\lfloor 5} - \ . \ . \ . \text{ etc.}$$

for purposes of comparison.

The first thing to be noticed is that the cosine and sine series are simpler for imaginary angles than for real, and special names have been given to these simpler series, the first being called the *hyperbolic cosine of* θ, and the second (without the i in front of it) the *hyperbolic sine of* θ, which leads to two equations, which may be taken equally well for definitions of the hyperbolic functions,

$$\text{hyp. cosine } \theta = \text{cosine } i\theta,$$
$$\text{hyp. sine } \theta = -i \text{ sine } i\theta.$$

Hyperbolic cosines and sines lead to a complete hyperbolic trigonometry parallel with ordinary trigonometry but having differences which perplex the memory. It is therefore an additional advantage in the use of imaginary quantities that their use amounts to including hyperbolic trigonometry in common trigonometry, which is much simpler than learning and using it separately. Had trigonometry been developed from the analytical standpoint, it is likely that it might have had as its basis the hyperbolic cosine and sine as simpler than the circular.

INTERRELATIONS.

Having now these generalized and extended views of the fundamental quantities i, ε^x, cosine x, and sine x, their relations to each other must be further examined, being of the greatest practical use in higher mathematics.

Rearranging the series last given, there results

$$\varepsilon^{i\theta} = 1 - \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}} - \frac{\theta^6}{\underline{|6}} + \ldots \text{ etc.}$$

$$+ i\left(\theta - \frac{\theta^3}{\underline{|3}} + \frac{\theta^5}{\underline{|5}} - \frac{\theta^7}{\underline{|7}} + \ldots \text{ etc.}\right)$$

which being compared with the defining series for cosine and sine gives

$$\varepsilon^{i\theta} = \text{cosine } \theta + i \text{ sine } \theta,$$

and in the same way it is found that

$$\varepsilon^{-i\theta} = \text{cosine } \theta - i \text{ sine } \theta,$$

which also comes directly from the previous formula, because changing the algebraic sign of an angle changes the algebraic sign of its sine.

These values for $\varepsilon^{i\theta}$ and $\varepsilon^{-i\theta}$ are complex quantities and may be laid off as shown in Figure 5 and Figure 6. The cosine being real is to be laid off as an x co-ordinate from the origin along the $X=$ axis, and at its end the imaginary quantity i sine θ is to be laid off as the y co-ordinate, plus in one case and minus in the other. Evidently then $\varepsilon^{i\theta}$, as previously stated, will always be on the unit circle, and to find a point representing $r\varepsilon^{i\theta}$ is simply to increase the radius from unity to r, so that any point in the plane can be expressed by $r\varepsilon^{i\theta}$ or $\varepsilon^{\log r + i\theta}$ as shown in the foregoing figures. This amounts to a sort of exponential trigonometry.

This representation of any point in the co-ordinate plane by exponentials leads to a conception either of the point itself, or of the radius from the origin to the point as a single quantity, which may be represented by a single letter, thus,

$z = \varepsilon^{\log r + i\theta} = r\varepsilon^{i\theta}$, and this conception is justified by the fact that this z can be put into an equation and subjected to algebraic treatment like any other quantity without error.

By addition and subtraction of the last formulæ exponential values are arrived at for cosine and sine which are highly important, and should be memorized.

$$\cos\theta = \frac{\varepsilon^{i\theta} + \varepsilon^{-i\theta}}{2},$$

$$\sin\theta = \frac{\varepsilon^{i\theta} - \varepsilon^{-i\theta}}{2i},$$

the corresponding formulæ for hyperbolic functions being

$$\text{hyp. } \cos\theta = \frac{\varepsilon^{\theta} + \varepsilon^{-\theta}}{2},$$

$$\text{hyp. } \sin\theta = \frac{\varepsilon^{\theta} - \varepsilon^{-\theta}}{2},$$

which again emphasizes the greater simplicity of these functions.

From the same two formulæ are obtained

$$\log(\cos\theta + i\sin\theta) = i\theta,$$
$$\log(\cos\theta - i\sin\theta) = -i\theta.$$

In the following pages it will be shown how these principles of higher mathematics apply to simple integration.

INTEGRATION.

The integration of algebraic forms containing the radical $\sqrt{\pm(x^2 + 2ax + b)}$, a and b having any values whatever, depends on the removal of the radical sign, which can be done

by imaginary and trigonometric substitution. As physical problems often depend upon angles, the change to a trigonometric form is likely to simplify the problem, and often in a marked degree, besides furnishing interesting and valuable practice in the principles developed. Thus,

putting $x = u - a$ the radical reduces to

$$\sqrt{u^2 \pm c^2} \text{ where } c^2 = b - a^2 \text{ is } \pm$$

according to the sign and magnitudc of b. Both signs can now be changed together if desired, by multiplying the radical by i and dividing inside by -1, which gives

$$i\sqrt{-u^2 \mp c^2},$$

and then the substitution $u = iv$ will change the sign of u^2 and give, if desired,

$$i\sqrt{+v^2 \mp c^2},$$

so that the original radical can be reduced to any one of three forms,

$$\sqrt{x^2 + c^2}, \qquad \sqrt{x^2 - c^2}, \qquad \text{or} \quad \sqrt{c^2 - x^2},$$

and these radicals can be removed by substituting $x = c \tan \theta$ in the first, $x = c \sec \theta$ in the second, and $x = c \cos \theta$ or $c \sin \theta$ in the third. By the imaginary factor therefore the signs can be changed at will so as to make use of the particular trigonometric substitution desired; and the fact that the differential coefficients of cosine and sine are of the first power, and those of the tangent and secant of the second power, makes it possible by this choice to change the exponents in the resulting form.

If the original expression is or leads to one of the simpler forms, as

$$dy = \frac{dx}{\sqrt{c^2 - x^2}} = \frac{d\left(\frac{x}{c}\right)}{\sqrt{1 - \left(\frac{x}{c}\right)^2}}$$

where the quantities may be real or imaginary, there is no necessity for trigonometric substitution, though it is interesting to make such for purpose of comparison, and the integration can be made directly into a circular function, and thereby practice be had in the use of imaginary arcs and with the exponential values of cosine and sine, and with constants of integration.

Much can be learned by integrating the same expression in various ways; the result may appear in different forms, and it is excellent practice to reduce one form to another, which can always be done if integrated correctly. In such a reduction the constant of integration plays an important part.

It remains then to consider the integration of simple trigonometric forms, which will be classified and discussed in a general but orderly manner. It is not intended that teacher or student shall follow this classification; indeed, it is better to avoid all classification, as more will be learned by solving each problem independently according to the general principles given. Many problems can be solved by several methods, and that given in the classification may not always be the simplest. Then again a different classification might be made, and more is learned by making or attempting one than by following one already made.

But this classification will nevertheless be of use, for the fact of having one that covers the whole ground (those in text-books do not always do so) is proof that all the forms can be

integrated by the principles given; and further, if some of them are attempted without success, the classification can then be consulted. Of course, without any classification it can be seen that all such forms may be integrated, for the exponential values of cosine and sine may be substituted, when the expression will simplify into a series of exponential terms of various powers which integrate at once, but this is not an easy way for many problems.

CLASSIFICATION.

Any product of integral powers of simple trigonometric functions can be put in the form

$$\cos^m x \sin^n x \, dx, \qquad m, \text{ an integer} \gtreqless n, \text{ an integer.}$$

The possibility of cases appearing in the analysis where the result depends on whether $m >$ or $< n$ might suggest a provision against such ambiguity by supposing at once $m > n$. This would necessitate an equal number of cases of the *complementary form*,

$$\sin^m x \cos^n x \, dx,$$

which is equivalent to $m < n$ in the original form.

However, this complementary form may be disposed of as follows:

Put $(x+y) = \frac{\pi}{2}$, then $\cos x = \sin y$, $\sin x = \cos y$, $dx = -dy$,

and we get, $\sin^m x \cos^n x \, dx = -\cos^m y \sin^n y \, dy$,

which reduces the complementary to the *standard form* with the sign changed; so that the integral of the complementary form can be obtained by integrating the corresponding standard form and changing the sign. If the integral be obtained in this and some other way, the constants of integration may need adjustment to make the results agree.

The standard form,

$$\cos^m x \sin^n x\, dx, \qquad m, \text{ integral and } \gtreqless n, \text{ integral},$$

therefore, includes all forms, which must belong to one of the sixteen cases symmetrically arranged in four groups in the following table:

			m			
			+		−	
			odd	even	odd	even
n	+	odd	a	a	a	a
		even	a	c	d	c
	−	odd	a	d	b	d
		even	a	c	d	b

The groups a, b, c and d will now be considered separately.

Group a.

One exponent + odd, the other $\pm$ odd or $\pm$ even.

$$\cos^m x \sin^n x\, dx = \sin^n x \cos^{2p} x \cos x\, dx,$$

if $\quad m$ is $+$odd and $= 2p+1$,

$$= \sin^n x\, (1 - \sin^2 x)^p\, d\,(\sin x)$$

$$= (\sin^n x - p \sin^{n+2} x + q \sin^{n+4} x - \ldots \text{ etc.})\, d\,(\sin x)$$

where the terms of the series integrate as powers of $\sin x$;

or; $\quad \cos^m x \sin^n x\, dx = \cos^m x \sin^{2p} x \sin x\, dx,$

if $\quad n$ is $+$odd and $= 2p+1$,

$$= -\cos^m x\, (1 - \cos^2 x)^p\, d\,(\cos x)$$

$$= -(\cos^m x - p \cos^{m+2} x + q \cos^{m+4} x - \ldots \text{ etc.})\, d\,(\cos x),$$

where the terms integrate as powers of the cosine. This second supposition is equivalent to the complementary form.

Group b.

Both exponents $-$odd or both $-$even.

For convenience put the sum of the exponents $= -2p$, we shall now always have $2p +$ even.

$$\cos^m x \sin^n x\, dx = \frac{\sin^n x \cos^m x \cos^n x\, dx}{\cos^n x}$$
$$= \tan^n x \cos^{m+n} x\, dx = \tan^n x \sec^{2p-2} x \sec^2 x\, dx$$
$$= \tan^n x\, (1 + \tan^2 x)^{p-1}\, d(\tan x)$$
$$= \{\tan^n x + (p-1) \tan^{n+2} x + q \tan^{n+4} x, \text{ etc.}\}\, d(\tan x),$$

the terms of which integrate as powers of the tangent. These powers start as minus powers, but may become plus toward the end of the series. It evidently makes no difference whether $m \gtreqless n$.

Group c.

Both exponents even, one $+$even and the other $\pm$ even.

Let the exponent which is plus even $= 2p$,

$$\cos^m x \sin^n x\, dx = \cos^m x\, (1 - \cos^2 x)^p\, dx, \qquad \text{if } n = 2p,$$
$$= (\cos^m x - p \cos^{m+2} x + q \cos^{m+4} x, \text{ etc.})\, dx,$$

or; $= \sin^n x\, (1 - \sin^2 x)^p\, dx, \qquad \text{if } m = 2p,$

$$= (\sin^n x - p \sin^{n+2} x + q \sin^{n+4} x, \text{ etc.})\, dx.$$

If both m and n are plus, the exponents of the terms of the series will be plus, but if one is minus there will be minus exponents in the series. All the exponents will, however, be even.

Terms with plus exponents integrate by doubling of

the angle, as often as may be necessary, by means of the formulæ

$$\cos^2 x = \frac{1+\cos 2x}{2}, \qquad \sin^2 x = \frac{1-\cos 2x}{2}.$$

Terms with minus exponents fall under group b by letting the other exponent equal 0.

When $m+n$ is $-$even, the method of group b can also be employed.

Group d.

One exponent $-$odd, the other $\pm$ even.

By the use of one of the following imaginary trigonometric substitutions the $-$odd exponent can be changed to plus.

Put $\sin x = i \tan \theta$ and there results the following set of values:

$$\begin{aligned} \cos x &= \sec \theta, & \sec x &= \cos \theta, \\ \sin x &= i \tan \theta, & \csc x &= -i \cot \theta, \\ \tan x &= i \sin \theta, & \cot x &= -i \csc \theta; \\ dx &= i \sec \theta \, d\theta, \end{aligned}$$

and for $\cos x = i \cot \theta$ there result the complementary substitutions:

$$\begin{aligned} \sin x &= \csc \theta, & \csc x &= \sin \theta, \\ \cos x &= i \cot \theta, & \sec x &= -i \tan \theta, \\ \cot x &= i \cos \theta, & \tan x &= -i \sec \theta. \\ dx &= i \csc \theta \, d\theta, \end{aligned}$$

The first gives

$$\begin{aligned} \cos^m x \sin^n x \, dx &= \sec^m \theta \, (i \tan \theta)^n \, i \sec \theta \, d\theta \\ &= \pm i \cos^{-(m+n+1)}\theta \sin^n \theta \, d\theta, \end{aligned}$$

and the second gives

$$\begin{aligned} \cos^m x \sin^n x \, dx &= (i \cot \theta)^m \csc^n \theta \, i \csc \theta \, d\theta \\ &= \pm i \cos^m \theta \sin^{-(m+n+1)}\theta \, d\theta. \end{aligned}$$

As $m+n+1$ is always even, the first should be used when m is odd, and the second when n is odd. Both exponents are now even, and, as can easily be seen, one will always be plus, which places them in group c.

The following use of the Imaginary is interesting on account of the mechanical analogy involved.

To integrate $$du = \varepsilon^x \cos x\, dx;$$

add to it the complementary equation

$$dv = \varepsilon^x \sin x\, dx$$

multiplied by the Imaginary Unit i and get

$$du + i\,dv = \varepsilon^x (\cos x + i \sin x)\, dx, \qquad \text{or}$$
$$d(u + iv) = dz = \varepsilon^x \varepsilon^{ix}\, dx = \varepsilon^{(1+i)x} dx.$$

dz is here a Complex Quantity, but in integrating it the real and imaginary parts must integrate separately and form the real and imaginary parts of the integral. The result is therefore

$$z = u + iv = \frac{1}{1+i}\,\varepsilon^{(1+i)x} + C$$
$$= \frac{1-i}{2}\,\varepsilon^x (\cos x + i \sin x) + C.$$

Separating the real and imaginary parts

$$u = \tfrac{1}{2}\,\varepsilon^x (\cos x + \sin x) + C_1$$
$$iv = \tfrac{1}{2}\,\varepsilon^x i (\sin x - \cos x) + C_2, \qquad \text{or}$$
$$v = \tfrac{1}{2}\,\varepsilon^x (\sin x - \cos x) + C_3.$$

The given equation represents a periodic motion along the Real Axis and the equation added represents one along the Imaginary Axis of an equal and similar degree of complexity,

but the combined equation represents a simple spiral motion around the origin with the variable radius ε^x, with an equally simple integration. The simplest form of this mechanical analogy is the combination of two mutually rectangular harmonic motions to form a uniform motion in a circle.

Or let $$d^2u = \varepsilon^x \sin x\, dx^2$$
and add $$id^2v = i\varepsilon^x \cos x\, dx^2;$$
then $$d^2z = d^2(u+iv) = i\varepsilon^x (\cos x - i \sin x)\, dx^2$$
$$= i\varepsilon^{(1-i)x}\, dx^2.$$

Integrating, there results
$$\frac{1-i}{i}\, dz = (\varepsilon^{(1-i)x} + C)\, dx.$$

Integrating again
$$\frac{(1-i)^2}{i}\, z = \varepsilon^{(1-i)x} + C_1 x + C_2\ ,$$
where $$C_1 = C\,(1-i)\,,$$
or $$-2z = -2\,(u+iv) = \varepsilon^x (\cos x - i \sin x) + C_1 x + C_2.$$

Separation of the real and imaginary parts leads to
$$-2u = \varepsilon^x \cos x + C_3 x + C_4,$$
$$2v = \varepsilon^x \sin x + C_5 x + C_6.$$

It may readily be inferred that complex quantities are as important and necessary in mathematical as revolving bodies are in mechanical work and analogies between them are often helpful.

J. BURKITT WEBB.

HOBOKEN, N. J.,
January 1, 1907.

INTEGRATION BY TRIGONOMETRIC AND IMAGINARY SUBSTITUTION.

CHAPTER I.

TRIGONOMETRIC DIFFERENTIALS.

1. Trigonometric Formulæ.—The method of integration developed in the following pages is designed to replace the usual "reduction formulæ," and being founded upon trigonometry the more important trigonometric relations are, for convenience of reference, tabulated below.

1. $\cos x = \dfrac{1}{\sec x}, \qquad \sin x = \dfrac{1}{\csc x}, \qquad \tan x = \dfrac{1}{\cot x}.$

2. $\tan x = \dfrac{\sin x}{\cos x}, \qquad \cot x = \dfrac{\cos x}{\sin x}.$

3. $\cos\left(\dfrac{\pi}{2} - x\right) = \sin x,$ 4. $\cos^2 x + \sin^2 x = 1,$

$\sin\left(\dfrac{\pi}{2} - x\right) = \cos x, \qquad \sec^2 x - \tan^2 x = 1,$

$\tan\left(\dfrac{\pi}{2} - x\right) = \cot x \qquad \csc^2 x - \cot^2 x = 1.$

5. $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y.$

6. $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y.$

7. $\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}.$

8. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

9. $\sin 2x = 2 \sin x \cos x, \qquad \tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}.$

10. $\cos^2 x = \dfrac{1 + \cos 2x}{2}, \qquad \sin^2 x = \dfrac{1 - \cos 2x}{2}.$

11. $\cos x + \cos y = \quad 2 \cos \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y).$

12. $\cos x - \cos y = -2 \sin \frac{1}{2}(x+y) \sin \frac{1}{2}(x-y).$

13. $\sin x + \sin y = \quad 2 \sin \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y).$

14. $\sin x - \sin y = \quad 2 \cos \frac{1}{2}(x+y) \sin \frac{1}{2}(x-y).$

The student should be familiar with the following relations from his study of the Differential Calculus.*

$$\cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \dots \qquad (15)$$

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \dots \qquad (16)$$

$$\varepsilon^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \frac{x^5}{\underline{|5}} + \frac{x^6}{\underline{|6}} + \frac{x^7}{\underline{|7}} + \dots \qquad (17)$$

By substituting another variable, $i\theta$, $(i = \sqrt{-1})$, for x in (17) there results

$$\varepsilon^{i\theta} = 1 + i\theta - \frac{\theta^2}{\underline{|2}} - i\frac{\theta^3}{\underline{|3}} + \frac{\theta^4}{\underline{|4}} + i\frac{\theta^5}{\underline{|5}} - \frac{\theta^3}{\underline{|6}} - \dots$$

$$= \left(1 - \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}} - \dots\right) + i\left(\theta - \frac{\theta^3}{\underline{|3}} + \frac{\theta^5}{\underline{|5}} - \dots\right),$$

which compared with (15) and (16), gives

$$\varepsilon^{i\theta} = \cos\theta + i \sin\theta \qquad (18)$$

* See Price, Vol. I, Arts. 59 to 61; Chauvenet's Plane and Spherical Trigonometry, Chapter XIV; and Peirce's Plane and Spherical Trigonometry, Chapter VII.

Substituting $-i\theta$ for x in (17) there results similarly

$$\varepsilon^{-i\theta} = \cos\theta - i\sin\theta \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Since ε is the base of the Naperian system of logarithms, (18) and (19) may be written

$$i\theta = \log(\cos\theta + i\sin\theta) \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and

$$-i\theta = \log(\cos\theta - i\sin\theta) \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Adding (18) and (19) gives

$$\cos\theta = \frac{\varepsilon^{i\theta} + \varepsilon^{-i\theta}}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Subtracting (19) from (18) results in

$$\sin\theta = \frac{\varepsilon^{i\theta} - \varepsilon^{-i\theta}}{2i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Dividing (23) by (22) gives

$$\tan\theta = \frac{\varepsilon^{i\theta} - \varepsilon^{-i\theta}}{i(\varepsilon^{i\theta} + \varepsilon^{-i\theta})} = \frac{\varepsilon^{2i\theta} - 1}{i(\varepsilon^{2i\theta} + 1)} \quad . \quad . \quad . \quad (24)$$

2 Trigonometric Differentials.—Every differential expression consisting of the product of integral powers of trigonometric functions of one angle, multiplied by the differential of the angle, can be reduced to the form

$$dy = \cos^m x \sin^n x \, dx,$$

in which m and n are integers, even or odd, positive or negative, or zero.

It will now be shown how each of the different cases of this expression may be integrated.

3. To integrate $dy = \cos^m x \sin^n dx$ when either m or n is an odd positive integer, no matter what the other may be.

(*a*) Let $m = 2r + 1$, r being a positive integer,

then $dy = \cos^m x \sin^n x \, dx$

$$= \cos^{2r+1} x \sin^n x \, dx$$
$$= \sin^n x \, (1 - \sin^2 x)^r \cos x \, dx$$
$$= \{\sin^n x - r \sin^{n+2} x + \ldots \pm \sin^{n+2r} x\} d(\sin x), \text{ and}$$

$$y = \frac{\sin^{n+1} x}{n+1} - \frac{r \sin^{n+3} x}{n+3} + \ldots \pm \frac{\sin^{n+2r+1} x}{n+2r+1} + C.$$

(*b*) Similarly, when n is of the form $2r + 1$,

$$dy = \cos^m x \sin^n x \, dx$$
$$= \cos^m x \sin^{2r+1} x \, dx$$
$$= \cos^m x \, (1 - \cos^2 x)^r \sin x \, dx$$
$$= -\{\cos^m x - r \cos^{m+2} x + \ldots \pm \cos^{m+2r} x\} d(\cos x), \text{ and}$$

$$y = -\frac{\cos^{m+1} x}{m+1} + \frac{r \cos^{m+3} x}{m+3} - \ldots \mp \frac{\cos^{m+2r+1} x}{m+2r+1} + C.$$

Example 1. To integrate $dy = \sin^2 x \cos^3 x \, dx$.

$$dy = \sin^2 x \, (1 - \sin^2 x) \, d(\sin x),$$
$$y = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

Ex. 2. To integrate $dy = \tan^3 x \, dx$.

$$dy = \frac{\sin^3 x \, dx}{\cos^3 x} = \frac{(1 - \cos^2 x)}{\cos^3 x} \sin x \, dx$$
$$= -\frac{d(\cos x)}{\cos^3 x} + \frac{d(\cos x)}{\cos x},$$

$$y = \frac{1}{2 \cos^2 x} + \log \cos x + C, \text{ or}$$
$$= \frac{\sec^2 x}{2} - \log \sec x + C.$$

EXAMPLES. I.

1. $dy = \sin x \cos x \, dx.$
2. $dy = \sin^3 x \, dx.$
3. $dy = \tan x \, dx.$
4. $dy = \cot x \, dx.$
5. $dy = \sin^2 x \cos^5 x \, dx.$
6. $dy = \cot^3 x \, dx.$
7. $dy = \sin^7 x \, dx.$
8. $dy = \cos^3 x \, dx.$
9. $dy = \dfrac{\sin^3 x \, dx}{\cos x}.$
10. $dy = \cos^7 x \, dx.$
11. $dy = \cot^7 x \, dx.$
12. $dy = \tan^5 x \, dx.$
13. $dy = \tan^3 x \sec x \, dx.$
14. $dy = \tan^3 x \sec^3 x \, dx.$
15. $dy = \dfrac{\sin^5 x \, dx}{\cos^3 x}.$
16. $dy = \dfrac{\cos^3 x \, dx}{\sin^2 x}.$

4. To integrate $dy = \cos^m x \sin^n x \, dx$ when $m + n$ is an even negative integer.

(*a*) Let $m + n = -2r$, then

$$
\begin{aligned}
dy &= \cos^m x \sin^n x \, dx \\
&= \tan^n x \cos^{m+n} x \, dx \\
&= \tan^n x \sec^{2r} x \, dx \\
&= \tan^n x \sec^{2r-2} x \sec^2 x \, dx \\
&= \tan^n x \, (1 + \tan^2 x)^{r-1} \, d\,(\tan x) \\
&= \{ \tan^n x + (r-1) \tan^{n+2} x + . \, . + \tan^{n+2r-2} x \} \, d\,(\tan x),
\end{aligned}
$$

and,

$$y = \frac{\tan^{n+1} x}{n+1} + (r-1) \frac{\tan^{n+3} x}{n+3} + . \; . \; . + \frac{\tan^{n+2r-1} x}{n+2r-1} + C.$$

(*b*) This integration may also be performed as follows:

$$
\begin{aligned}
dy &= \cos^m x \sin^n x \, dx \\
&= \cot^m x \sin^{m+n} x \, dx \\
&= \cot^m x \csc^{2r} x \, dx \\
&= -\cot^m x \, (1 + \cot^2 x)^{r-1} \, d\,(\cot x) \\
&= -\{ \cot^m x + (r-1) \cot^{m+2} x + . \, . + \cot^{m+2r-2} x \} \, d\,(\cot x),
\end{aligned}
$$

and $y = -\dfrac{\cot^{m+1} x}{m+1} - (r-1) \dfrac{\cot^{m+3} x}{m+3} - . \; . - \dfrac{\cot^{m+2r-1} x}{m+2r-1} + C.$

Example 1. To integrate $dy = \dfrac{dx}{\sin x \cos^3 x}$.

$$dy = \frac{\sec^4 x\, dx}{\tan x} = \frac{(1+\tan^2 x)\, d\,(\tan x)}{\tan x}$$

$$= \left(\frac{1}{\tan\ x} + \tan x\right) d(\tan x), \text{ and}$$

$$y = \log \tan\ x + \frac{\tan^2 x}{2} + C.$$

EXAMPLES. II.

1. $dy = \sec^4 x\, dx.$
2. $dy = \sec^6 x\, dx.$
3. $dy = \csc^4 x\, dx.$
4. $dy = \csc^6 x\, dx.$
5. $dy = \dfrac{dx}{\sin x \cos x}.$
6. $dy = \dfrac{dx}{\sin^2 x \cos^2 x}.$
7. $dy = \dfrac{\sin^2 x\, dx}{\cos^4 x}.$
8. $dy = \cot^2 x \csc^6 x\, dx.$
9. $dy = \tan^3 x \sec^4 x\, dx.$
10. $dy = \dfrac{dx}{\sin^2 x \cos^4 x}.$

5. To integrate $dy = \cos^m x \sin^n x\, dx$ when both m and n are even positive integers. When both m and n are even positive integers, by doubling the angle as often as necessary by means of the following trigonometric relations,

$$\sin x \cos x = \frac{\sin 2x}{2} \quad . \quad . \quad . \quad . \quad . \quad (9, \text{ p. } 28)$$

$$\cos^2 x = \frac{1+\cos 2x}{2} \quad . \quad . \quad . \quad (10, \text{ p. } 28)$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \quad . \quad . \quad . \quad (10, \text{ p. } 28)$$

this differential expression may be transformed into an expression involving sines and cosines of multiple angles and then integrated.

This may be best illustrated by means of a few examples.

EXAMPLE 1. To integrate $dy = \sin^2 x\, dx$.

$$dy = \frac{1 - \cos 2x}{2}\, dx,$$

$$y = \frac{x}{2} - \frac{\sin 2x}{4} + C.$$

Ex. 2. To integrate $dy = \sin^2 x \cos^2 x\, dx$.

$$dy = \frac{\sin^2 2x}{4}\, dx$$

$$= \frac{1 - \cos 4x}{8}\, dx,$$

$$y = \frac{x}{8} - \frac{\sin 4x}{32} + C.$$

Ex. 3. To integrate $dy = \sin^2 x \cos^4 x\, dx$.

$$dy = \sin^2 x \cos^2 x \cos^2 x\, dx$$

$$= \frac{\sin^2 2x}{4}\left(\frac{1 + \cos 2x}{2}\right) dx$$

$$= \left(\frac{\sin^2 2x}{8} + \frac{\sin^2 2x \cos 2x}{8}\right) dx$$

$$= \left(\frac{1 - \cos 4x}{16} + \frac{\sin^2 2x \cos 2x}{8}\right) dx,$$

$$y = \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C.$$

EXAMPLES. III.

1. $dy = \cos^2 x\, dx$.

2. $dy = \sin^4 x\, dx$.

3. $dy = \cos^4 x\, dx$.

4. $dy = \sin^4 x \cos^2 x\, dx$.

5. $dy = \sin^4 x \cos^4 x\, dx$.

6. It will now be shown how the remaining cases of $dy = \cos^m x \sin^n x \, dx$ may be integrated by one or more of the preceding cases, either directly or by means of an imaginary trigonometric substitution.

7. **To integrate $dy = \cos^m x \sin^n x \, dx$ when $m + n$ is an odd negative integer.**

I. *When either m or n is an even positive integer.*

(a). When n is an even positive integer.

Let $$n = 2p \text{ and } m = -(2p + 2r + 1),$$

then $$m + n = -(2r + 1), \text{ an odd negative integer,}$$

and
$$\begin{aligned} dy &= \cos^m x \sin^n x \, dx \\ &= \sin^{2p} x \cos^{-(2p+2r+1)} x \, dx \\ &= \tan^{2p} x \sec^{2r+1} x \, dx \\ &= \tan^{2p} x \sec^{2r-1} x \, d(\tan x) \quad . \quad . \quad . \quad . \quad (A) \end{aligned}$$

which last expression may now be transformed into

$$dy = (i)^{2p+1} \sin^{2p} \theta \cos^{2r} \theta \, d\theta$$

by means of the following imaginary trigonometric substitution:

Let $$\tan x = i \sin \theta, \text{ then } d(\tan x) = i \cos \theta \, d\theta,$$

and $$\sqrt{1 + \tan^2 x} = \sqrt{1 - \sin^2 \theta},$$

or, $$\sec x = \cos \theta.$$

Substituting these values in (A),

$$\begin{aligned} dy &= (i \sin \theta)^{2p} \cos^{2r-1} \theta \, i \cos \theta \, d\theta \\ &= (i)^{2p+1} \sin^{2p} \theta \cos^{2r} \theta \, d\theta \quad . \quad . \quad . \quad . \quad . \quad (B) \end{aligned}$$

to which last expression the method of Art. 5 is applicable, since $2p$ and $2r$ are even positive integers.

As far as the integration is concerned, i is treated the same as any other constant. After integration the functions of θ

are replaced by the corresponding functions of x, and $i\theta$ is replaced by log $(\cos\theta + i\sin\theta)$ and $-i\theta$ by log $(\cos\theta - i\sin\theta)$. [See p. 29, (20) and (21).]

EXAMPLE 1. To integrate $dy = \sec x\,dx$.

$$dy = \frac{\sec^2 x\,dx}{\sec x} = \frac{d\,(\tan x)}{\sec x}$$

$$= \frac{i\cos\theta\,d\theta}{\cos\theta} = i\,d\theta,$$

and
$$y = i\theta + C$$
$$= \log\,(\cos\theta + i\sin\theta) + C$$
$$= \log\,(\sec x + \tan x) + C.$$

Ex. 2. To integrate $dy = \tan^2 x\sec^3 x\,dx$.

$$dy = \tan^2 x\sec x\,d\,(\tan x)$$
$$= (i\sin\theta)^2\cos\theta\;i\cos\theta\,d\theta$$
$$= -i\sin^2\theta\cos^2\theta\,d\theta,$$

and $\quad y = -i\left(\dfrac{\theta}{8} - \dfrac{\sin 4\theta}{32}\right) + C \qquad$ (See Ex. 2, Art. 5.)

$$= \frac{1}{8}\log\,(\cos\theta - i\sin\theta) + \frac{i\sin\theta\cos\theta}{8}(1 - 2\sin^2\theta) + C$$

$$= \frac{1}{8}\log(\sec x - \tan x) + \frac{\tan x\sec x}{8}(1 + 2\tan^2 x) + C.$$

(b) When m is an even positive integer.

Let $\quad m = 2p$ and $n = -(2p + 2r + 1)$, then

$$dy = \cos^m x\sin^n x\,dx$$
$$= \cos^{2p} x\sin^{-(2p+2r+1)} x\,dx$$
$$= \cot^{2p} x\csc^{2r+1} x\,dx$$
$$= -\cot^{2p} x\csc^{2r-1} x\,d\,(\cot x)$$

which last expression is transformed into

$$dy = -(i)^{2p+1}\sin^{2p}\theta\cos^{2r}\theta\,d\theta \qquad . \quad . \quad . \quad (C)$$

by means of the following imaginary trigonometric substitution:

Let $\cot x = i \sin \theta$, then $d\ (\cot x) = i \cos \theta\ d\theta$,

and $\sqrt{1 + \cot^2 x} = \sqrt{1 - \sin^2 \theta}$,

or, $\csc x = \cos \theta$.

(C) is the same expression as (B) with change of sign, and can therefore also be integrated by method of Art. 5.

Ex. 3. To integrate $dy = \csc x\ dx$.

$$dy = \frac{\csc^2 x\ dx}{\csc x} = -\frac{d\ (\cot x)}{\csc x}$$
$$= -\frac{i \cos \theta\ d\theta}{\cos \theta} = -i\ d\theta,$$

and
$$y = -i\ \theta + C$$
$$= \log\ (\cos \theta - i \sin \theta) + C$$
$$= \log\ (\csc x - \cot x) + C.$$

Ex. 4. To integrate $dy = \cot^2 x \csc x\ dx$.

$$dy = -\frac{\cot^2 x\ d\ (\cot x)}{\csc x}$$
$$= -\frac{(i \sin \theta)^2\ i \cos \theta\ d\theta}{\cos \theta}$$
$$= i \sin^2 \theta\ d\theta$$
$$y = \frac{i\ \theta}{2} - \frac{i \sin 2\ \theta}{4} + C \qquad \text{(See Ex. 1, Art. 5.)}$$
$$= \frac{1}{2} \log\ (\cos \theta + i \sin \theta) - \frac{i \sin \theta \cos \theta}{2} + C$$
$$= \frac{1}{2} \log\ (\csc x + \cot x) - \frac{\cot x \csc x}{2} + C.$$

II. *When both m and n are negative integers.*

(a) When n is even and m odd.

Let $\quad n = -2p$ and $m = -(2r+1)$,

then $\quad m + n = -(2p + 2r + 1)$,

and
$$dy = \cos^m x \sin^n x\, dx$$
$$= \cos^{-(2r+1)} x \sin^{-2p} x\, dx$$
$$= \frac{dx}{\sin^{2p} x \cos^{2r+1} x} = \frac{\sec^{2p+2r+1} x\, dx}{\tan^{2p} x}$$
$$= \frac{\sec^{2p+2r-1} x\, d(\tan x)}{\tan^{2p} x}.$$

By use of the imaginary trigonometric substitution given in I (a),

$$dy = \frac{i \cos^{2p+2r} \theta\, d\theta}{(i \sin \theta)^{2p}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (D)$$
$$= (i)^{1-2p} \frac{(1 - \sin^2 \theta)^{p+r}}{\sin^{2p} \theta} d\theta$$
$$= (i)^{1-2p} \{\csc^{2p} \theta - (p+r) \csc^{2p-2} \theta + \;..\; \pm \sin^{2r} \theta\}\, d\theta,$$

each term of which can now be integrated by previous cases, since all the exponents are even; the terms involving $\csc \theta$ by Art. 4, and the terms involving $\sin \theta$ by Art. 5.

Ex. 5. To integrate $dy = \dfrac{dx}{\sin^2 x \cos x}$

$$dy = \frac{\sec^3 x\, dx}{\tan^2 x} = \frac{\sec x\, d(\tan x)}{\tan^2 x}$$
$$= \frac{i \cos^2 \theta\, d\theta}{(i \sin \theta)^2} = -i \frac{(1 - \sin^2 \theta)}{\sin^2 \theta} d\theta$$
$$= -i (\csc^2 \theta - 1)\, d\theta$$
$$y = i \cot \theta + i\, \theta + C$$
$$= -\frac{\cos \theta}{i \sin \theta} + \log(\cos \theta + i \sin \theta) + C$$
$$= -\frac{\sec x}{\tan x} + \log(\sec x + \tan x) + C$$
$$= \log(\sec x + \tan x) - \csc x + C.$$

(*b*) When m is even and n odd.

$$\begin{aligned}dy &= \cos^m x \sin^n x\, dx\\ &= \cos^{-2p} x \sin^{-(2r+1)} x\, dx\\ &= \frac{\csc^{2p+2r+1} x\, dx}{\cot^p x} = -\frac{\csc^{2p+2r-1} x\, d(\cot x)}{\cot^{2p} x}\end{aligned}$$

which by means of the imaginary trigonometric substitution given in I (*b*), is transformed into

$$dy = -\frac{i \cos^{2p+2r} \theta\, d\theta}{(i \sin \theta)^{2p}}$$

which last expression is the same as (*D*) with change of sign, and may therefore be integrated in the same way.

Ex. 6. To integrate $dy = \dfrac{dx}{\sin^3 x \cos^2 x}$.

$$\begin{aligned}dy &= \frac{\csc^5 x\, dx}{\cot^2 x} = -\frac{\csc^3 x\, d(\cot x)}{\cot^2 x}\\ &= -\frac{i \cos^4 \theta\, d\theta}{(i \sin \theta)^2} = \frac{i \cos^4 \theta\, d\theta}{\sin^2 \theta}\\ &= i\left(\frac{1 - 2 \sin^2 \theta + \sin^4 \theta}{\sin^2 \theta}\right) d\theta\\ &= i(\csc^2 \theta - 2 + \sin^2 \theta)\, d\theta\\ &= i\left(\csc^2 \theta - 2 + \frac{1}{2} - \frac{\cos 2\theta}{2}\right) d\theta\\ y &= -i \cot \theta - \frac{3}{2} i\theta - \frac{i \sin 2\theta}{4} + C\\ &= \frac{\cos \theta}{i \sin \theta} + \frac{3}{2} \log(\cos \theta - i \sin \theta) - \frac{i \sin \theta \cos \theta}{2} + C\\ &= \frac{\csc x}{\cot x} + \frac{3}{2} \log(\csc x - \cot x) - \frac{\cot x \csc x}{2} + C\\ &= \frac{3}{2} \log(\csc x - \cot x) - \frac{\cot x \csc x}{2} + \sec x + C.\end{aligned}$$

EXAMPLES. IV.

1. $dy = \sec^3 x\, dx.$
2. $dy = \sec^5 x\, dx.$
3. $dy = \csc^3 x\, dx.$
4. $dy = \csc^5 x\, dx.$
5. $dy = \tan^2 x \sec x\, dx.$
6. $dy = \tan^4 x \sec x\, dx.$
7. $dy = \cot^2 x \csc^3 x\, dx.$
8. $dy = \cot^4 x \csc x\, dx.$
9. $dy = \dfrac{dx}{\sin^2 x \cos^3 x}.$
10. $dy = \dfrac{dx}{\sin x \cos^2 x}.$
11. $dy = \dfrac{dx}{\sin^4 x \cos x}.$
12. $dy = \dfrac{\sin^4 x\, dx}{\cos^7 x}.$

8. To integrate $dy = \cos^m \sin^n x\, dx$ when either m or n is an even positive integer, the other being a negative integer, and $m+n$ is a positive integer or zero.

(*a*) When n is an even positive integer.

Let $\qquad n = 2p$ and $m = -(2p-r)$,

then $\qquad m+n = r$, a positive integer,

and
$$\begin{aligned} dy &= \cos^m x \sin^n x\, dx \\ &= \cos^{-(2p-r)} x \sin^{2p} x\, dx \\ &= \frac{(1-\cos^2 x)^p\, dx}{\cos^{2p-r} x} \\ &= (\sec^{2p-r} x - p \sec^{2p-r-2} x + \ldots \pm \cos^r x)\, dx \quad . \quad (E) \end{aligned}$$

(*b*) Similarly, when m is an even positive integer,

$$\begin{aligned} dy &= \cos^m x \sin^n x\, dx \\ &= \frac{(1-\sin^2 x)^p\, dx}{\sin^{2p-r} x} \\ &= (\csc^{2p-r} x - p \csc^{2p-r-2} x + \ldots \pm \sin^r x)\, dx \quad . \quad (F) \end{aligned}$$

(E) and (F) are similar expressions in which the exponents of all the terms are even or odd, according as r is even or odd. When r is even, the methods of Arts. 4 and 5 are employed, and when r is odd, Arts. 3 and 7.

EXAMPLE 1. To integrate $dy = \dfrac{\cos^2 x\, dx}{\sin x}$.

$$dy = \frac{(1-\sin^2 x)}{\sin x}\,dx$$

$$= (\csc x - \sin x)\,dx$$

$$y = \log(\csc x - \cot x) + \cos x + C. \qquad \text{(See Ex. 3, Art. 7.)}$$

Ex. 2. To integrate $dy = \tan^4 x\,dx$.

$$dy = \frac{\sin^4 x\,dx}{\cos^4 x} = \frac{(1 - 2\cos^2 x + \cos^4 x)}{\cos^4 x}\,dx$$

$$= (\sec^4 x - 2\sec^2 x + 1)\,dx$$

$$= (1 + \tan^2 x)\,d(\tan x) - 2\,d(\tan x) + dx.$$

$$y = \tan x + \frac{\tan^3 x}{3} - 2\tan x + x + C$$

$$= x - \tan x + \frac{\tan^3 x}{3} + C$$

EXAMPLES. V.

1. $dy = \tan^2 x\,dx$.
2. $dy = \dfrac{\sin^2 x\,dx}{\cos x}$.
3. $dy = \tan^4 x\,dx$.
4. $dy = \cot^2 x\,dx$.
5. $dy = \cot^6 x\,dx$.
6. $dy = \dfrac{\cos^4 x\,dx}{\sin^3 x}$.
7. $dy = \dfrac{\sin^4 x\,dx}{\cos^3 x}$.

9. To integrate

$$dy = \cos mx \cos nx \, dx, \quad dy = \sin mx \cos nx \, dx,$$

and $dy = \sin mx \sin nx \, dx$.

$$dy = \cos mx \cos nx \, dx$$

$$= [\tfrac{1}{2} \cos (m+n) x + \tfrac{1}{2} \cos (m-n) x] \, dx \qquad (11, \text{ p. } 28)$$

$$y = \frac{\sin (m+n) x}{2 (m+n)} + \frac{\sin (m-n) x}{2 (m-n)} + C.$$

$$dy = \sin mx \cos nx \, dx$$

$$= [\tfrac{1}{2} \sin (m+n) x + \tfrac{1}{2} \sin (m-n) x] \, dx \qquad (13, \text{ p. } 28)$$

$$y = -\frac{\cos (m+n) x}{2 (m+n)} - \frac{\cos (m-n) x}{2 (m-n)} + C.$$

$$dy = \sin mx \sin nx \, dx$$

$$= [-\tfrac{1}{2} \cos (m+n) x + \tfrac{1}{2} \cos (m-n) x] \, dx \qquad (12, \text{ p. } 28)$$

$$y = -\frac{\sin (m+n) x}{2 (m+n)} + \frac{\sin (m-n) x}{2 (m-n)} + C.$$

EXAMPLES. VI.

1. $dy = \cos 3x \sin 5x \, dx$.

2. $dy = \sin 5x \sin 6x \, dx$.

3. $dy = \cos 4x \cos 7x \, dx$.

4. $dy = \sin \frac{2}{3}x \cos \frac{4}{3}x \, dx$. (See 14, p. 28.)

5. $dy = \cos \frac{3}{4}x \sin \frac{1}{4}x \, dx$.

6. $dy = \cos 3x \cos \frac{4}{3}x \, dx$.

10. To integrate $dy = \epsilon^{ax}\cos bx\,dx$.*

Substituting for cos bx its value in exponential form [(22), p. 29],

$$dy = \varepsilon^{ax}\left(\frac{\varepsilon^{ibx} + \varepsilon^{-ibx}}{2}\right)dx = \frac{1}{2}\left[\varepsilon^{(a+ib)x} + \varepsilon^{(a-ib)x}\right]dx,$$

$$y = \frac{1}{2}\left[\frac{\varepsilon^{(a+ib)x}}{a+ib} + \frac{\varepsilon^{(a-ib)x}}{a-ib}\right] + C$$

$$= \frac{\varepsilon^{ax}}{2}\left[\frac{\varepsilon^{ibx}}{a+ib} + \frac{\varepsilon^{-ibx}}{a-ib}\right] + C$$

$$= \frac{\varepsilon^{ax}}{2}\left[\frac{a\varepsilon^{ibx} - ib\varepsilon^{ibx} + a\varepsilon^{-ibx} + ib\varepsilon^{-ibx}}{a^2+b^2}\right] + C$$

$$= \frac{\varepsilon^{ax}}{a^2+b^2}\left[\frac{a\,(\varepsilon^{ibx} + \varepsilon^{-ibx})}{2} - \frac{ib\,(\varepsilon^{ibx} - \varepsilon^{-ibx})}{2}\right] + C$$

$$= \frac{\varepsilon^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C. \qquad [(22) \text{ and } (23), \text{ p. } 29]$$

This last expression admits of further simplification by substituting

cos ω for $\dfrac{a}{\sqrt{a^2+b^2}}$, and sin ω for $\dfrac{b}{\sqrt{a^2+b^2}}$, where $\omega = \tan^{-1}\dfrac{b}{a}$,

hence $$y = \frac{\varepsilon^{ax}}{\sqrt{a^2+b^2}}(\cos\omega\cos bx + \sin\omega\sin bx) + C$$

$$= \frac{\varepsilon^{ax}}{\sqrt{a^2+b^2}}\cos(bx-\omega) + C \quad . \quad . \quad . \quad . \quad . \quad (5, \text{ p. } 27)$$

* Price, Vol. II, Art. 76.

11. To integrate $dy = \epsilon^{ax} \sin bx\, dx$.

Substituting for $\sin bx$, its value in exponential form [(23), p. 29],

$$dy = \varepsilon^{ax}\left(\frac{\varepsilon^{ibx} - \varepsilon^{-ibx}}{2i}\right)dx = \frac{1}{2i}\left[\varepsilon^{(a+ib)x} - \varepsilon^{(a-ib)x}\right]dx$$

$$y = \frac{1}{2i}\left[\frac{\varepsilon^{(a+ib)x}}{a+ib} - \frac{\varepsilon^{(a-ib)x}}{a-ib}\right] + C$$

$$= \frac{\varepsilon^{ax}}{2i}\left[\frac{\varepsilon^{ibx}}{a+ib} - \frac{\varepsilon^{-ibx}}{a-ib}\right] + C$$

$$= \frac{\varepsilon^{ax}}{2i}\left[\frac{a\varepsilon^{ibx} - ib\varepsilon^{ibx} - a\varepsilon^{-ibx} - ib\varepsilon^{-ibx}}{a^2+b^2}\right] + C$$

$$= \frac{\varepsilon^{ax}}{a^2+b^2}\left[\frac{a\,(\varepsilon^{ibx} - \varepsilon^{-ibx})}{2i} - \frac{ib\,(\varepsilon^{ibx} + \varepsilon^{-ibx})}{2i}\right] + C$$

$$= \frac{\varepsilon^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + C. \quad [(22) \text{ and } (23), \text{ p. } 29]$$

$$= \frac{\varepsilon^{ax}}{\sqrt{a^2+b^2}}(\cos\omega \sin bx - \sin\omega \cos bx) + C$$

$$= \frac{\varepsilon^{ax}}{\sqrt{a^2+b^2}}\sin(bx - \omega) + C, \text{ where } \omega = \tan^{-1}\frac{b}{a}. \quad (6, \text{ p. } 27)$$

12. The integration of the differential expressions of the two preceding articles may also be performed as follows:

Let $\qquad du = \varepsilon^{ax} \cos bx\, dx$ and $dv = \varepsilon^{ax} \sin bx\, dx,$

multiplying the latter through by i, and adding to the former

$$du + idv = \varepsilon^{ax} \cos bx\, dx + i\, \varepsilon^{ax} \sin bx\, dx$$

$$= \varepsilon^{ax} (\cos bx + i \sin bx)\, dx$$

$$= \varepsilon^{ax}\, \varepsilon^{ibx}\, dx = \varepsilon^{(a+ib)x}\, dx \quad . \quad . \quad . \quad (18,\ p.\ 28)$$

$$u + iv = \frac{\varepsilon^{(a+ib)x}}{a+ib} + C$$

$$= \frac{a - ib}{a^2 + b^2}\, \varepsilon^{ax}\, \varepsilon^{ibx} + C$$

$$= \frac{a - ib}{a^2 + b^2}\, \varepsilon^{ax} (\cos bx + i \sin bx) + C$$

$$= \frac{\varepsilon^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$$

$$+ \frac{i\varepsilon^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + C.$$

Equating the real parts,

$$u = \frac{\varepsilon^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + C'$$

$$= \frac{\varepsilon^{ax}}{\sqrt{a^2 + b^2}} \cos \left(bx - \tan^{-1} \frac{b}{a}\right) + C'$$

Equating the imaginary parts and dividing through by i,

$$v = \frac{\varepsilon^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + C_1$$

$$= \frac{\varepsilon^{ax}}{\sqrt{a^2+b^2}} \sin\left(bx - \tan^{-1}\frac{b}{a}\right) + C_1.$$

EXAMPLES. VII.

1. $dy = \varepsilon^x \cos x \, dx.$

2. $dy = \varepsilon^x \sin x \, dx.$

3. $dy = \varepsilon^{2x} \cos 3x \, dx.$

4. $dy = \dfrac{\sin x \, dx}{\varepsilon^x}.$

5. $dy = \varepsilon^{-3x} \cos 2x \, dx.$

Find the value of I in the following equations:

6. $I\varepsilon^{\frac{R}{L}t} = \dfrac{e_m}{L}\displaystyle\int \varepsilon^{\frac{R}{L}t} \sin \omega t \, dt + k.$

7. $I\varepsilon^{\frac{t}{RC}} = \dfrac{\omega e_m}{R}\displaystyle\int \varepsilon^{\frac{t}{RC}} \cos \omega t \, dt + k.$

MISCELLANEOUS EXAMPLES. I.

1. $dy = \sin^5 x\, dx.$
2. $dy = \sin^6 x\, dx.$
3. $dy = \sin^3 x \cos^4 x\, dx.$
4. $dy = \tan x \sec^3 x\, dx.$
5. $dy = \tan^6 x\, dx.$
6. $dy = \tan^7 x\, dx.$
7. $dy = \cot^4 x\, dx.$
8. $dy = \cot^5 x\, dx.$
9. $dy = \sec^7 x\, dx.$
10. $dy = \sec^8 x\, dx.$
11. $dy = \cos^5 x\, dx.$
12. $dy = \cos^6 x\, dx.$
13. $dy = \csc^7 x\, dx.$
14. $dy = \csc^8 x\, dx.$
15. $dy = \tan^2 x \sec^5 x\, dx.$
16. $dy = \sin^6 x \cos^2 x\, dx.$
17. $dy = \dfrac{dx}{\sin^4 x \cos^2 x}.$
18. $dy = \dfrac{\cos^4 x\, dx}{\sin^8 x}.$
19. $dy = \dfrac{dx}{\sin x \cos^4 x}.$
20. $dy = \dfrac{\sin^5 x\, dx}{\sqrt{\cos x}}.$
21. $dy = \varepsilon^{2x} \cos 4x\, dx.$
22. $dy = \cos 4x \cos 5x\, dx.$
23. $dy = x^2 \sin^2 (x^3) \cos^3 (x^3)\, dx.$
24. $dy = \tan^5 x \sec^3 x\, dx.$
25. $dy = \dfrac{\sin^4 x\, dx}{\cos x}.$
26. $dy = \cot^4 x \csc^3 x\, dx.$
27. $dy = \cot^2 x \csc^5 x\, dx.$
28. $dy = \sin^2 x \cos^6 x\, dx.$
29. $dy = \dfrac{\cos^6 x\, dx}{\sin^5 x}.$
30. $dy = \dfrac{\cos^5 x\, dx}{\sqrt{\sin x}}.$
31. $dy = \cot x \csc^5 x\, dx.$
32. $dy = \varepsilon^{3x} \cos 4x\, dx.$

CHAPTER II.

RATIONALIZATION BY TRIGONOMETRIC SUBSTITUTION.*

13. Introduction. Since in a right-angled triangle the square of the hypothenuse is equal to the sum of the squares of the other two sides, $\sqrt{a^2+x^2}$ may be represented by the hypothenuse, and $\sqrt{a^2-x^2}$ or $\sqrt{x^2-a^2}$ by one of the other two sides of a right-angled triangle, and each one of these surds may therefore be expressed rationally as a trigonometric function of one of the acute angles of the triangle.

Differential expressions containing one of these surds may therefore be rationalized by transforming into trigonometric functions, and, unless otherwise too complicated, integrated by methods given in Chapter I.

14. Rationalization of expressions containing $\sqrt{a^2+x^2}$.

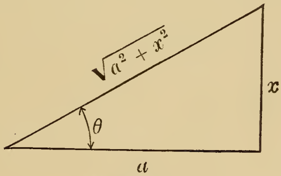


Fig. 1.

From the triangle in Fig. 1,

$$\sqrt{a^2+x^2} = a \sec \theta,$$

$$x = a \tan \theta,$$

and $$dx = a \sec^2 \theta \, d\theta.$$

* Price, Vol. II, Art. 79.

EXAMPLE 1. To integrate $dy = \dfrac{dx}{a^2 + x^2}$.

Substituting the values obtained above,

$$dy = \frac{a \sec^2 \theta \, d\theta}{a^2 \sec^2 \theta} = \frac{d\theta}{a},$$

$$y = \frac{\theta}{a} + C.$$

Substituting the values for θ obtained directly from the triangle in Fig. 1,

$$y = \frac{1}{a} \tan^{-1} \frac{x}{a} + C.$$

Ex. 2. To integrate $dy = \dfrac{dx}{(a^2 + x^2)^{\frac{3}{2}}}$.

$$dy = \frac{a \sec^2 \theta \, d\theta}{a^3 \sec^3 \theta} = \frac{d\theta}{a^2 \sec \theta} = \frac{\cos \theta \, d\theta}{a^2},$$

$$y = \frac{1}{a^2} \sin \theta + C$$

$$= \frac{x}{a^2 \sqrt{a^2 + x^2}} + C.$$

EXAMPLES. VIII.

1. $dy = \dfrac{x^3 \, dx}{(1 + x^2)^{\frac{5}{2}}}$.

2. $dy = \dfrac{dx}{(a^2 + x^2)^{\frac{7}{2}}}$.

3. $dy = \dfrac{dx}{x (a^2 + x^2)}$.

4. $dy = \dfrac{dx}{x^3 (a^2 + x^2)}$.

5. $dy = \dfrac{dx}{(a^2 + x^2)^{\frac{5}{2}}}$.

6. $dy = \dfrac{x^3 \, dx}{(a^2 + x^2)^2}$.

7. $dy = \dfrac{x^5 \, dx}{a^2 + x^2}$.

8. $dy = \dfrac{x^3 \, dx}{\sqrt{a^2 + x^2}}$.

9. $dy = x^3 \sqrt{1 + x^2} \, dx$.

10. $dy = \dfrac{x^4 \, dx}{(a^2 + x^2)^3}$.

11. $dy = \dfrac{x^5 \, dx}{(1 + x^2)^3}$.

12. $dy = \dfrac{x^2 \, dx}{(a^2 + x^2)^3}$.

15. Rationalization of expressions containing $\sqrt{a^2 - x^2}$.

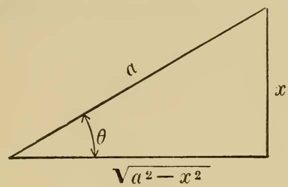


Fig. 2.

From the triangle in Fig. 2,

$$\sqrt{a^2 - x^2} = a \cos \theta,$$

$$x = a \sin \theta,$$

and

$$dx = a \cos \theta \, d\theta.$$

EXAMPLE 1. To integrate $dy = \dfrac{dx}{\sqrt{a^2 - x^2}}$.

$$dy = \frac{a \cos \theta \, d\theta}{a \cos \theta} = d\theta$$

$$y = \theta + C = \sin^{-1} \frac{x}{a} + C.$$

Ex. 2. To integrate $dy = \dfrac{x^3 \, dx}{\sqrt{a^2 - x^2}}$.

$$dy = \frac{a^3 \sin^3 \theta \, a \cos \theta \, d\theta}{a \cos \theta} = a^3 \sin^3 \theta \, d\theta$$

$$= a^3 (1 - \cos^2 \theta) \sin \theta \, d\theta,$$

$$y = -a^2 \cos \theta + \frac{a^3 \cos^3 \theta}{3} + C$$

$$= -\frac{a^3 \cos \theta}{3} (3 - \cos^2 \theta) + C$$

$$= -\frac{a^2 \sqrt{a^2 - x^2}}{3} \left(3 - \frac{a^2 - x^2}{a^2}\right) + C$$

$$= -\frac{\sqrt{a^2 - x^2}}{3} (2a^2 + x^2) + C.$$

EXAMPLES. IX.

1. $dy = \dfrac{x^2\,dx}{\sqrt{a^2 - x^2}}$.

2. $dy = \sqrt{a^2 - x^2}\,dx$.

3. $dy = \dfrac{x^5\,dx}{\sqrt{a^2 - x^2}}$.

4. $dy = (a^2 - x^2)^{\frac{3}{2}}\,dx$.

5. $dy = \dfrac{x^2\,dx}{(a^2 - x^2)^{\frac{3}{2}}}$.

6. $dy = \dfrac{x^3\,dx}{(a^2 - x^2)^2}$.

7. $dy = \dfrac{\sqrt{a^2 - x^2}\,dx}{x^2}$.

8. $dy = \dfrac{(1 - x^2)^{\frac{5}{2}}\,dx}{x^6}$

9. $dy = \dfrac{dx}{x^2\sqrt{a^2 - x^2}}$.

10. $dy = \dfrac{dx}{(a^2 - x^2)^{\frac{3}{2}}}$.

11. $dy = x^2\sqrt{a^2 - x^2}\,dx$.

12. $dy = x^2(1 - x^2)^{\frac{3}{2}}\,dx$.

13. $dy = \dfrac{dx}{x(a^2 - x^2)}$.

14 $dy = \dfrac{x^3\,dx}{(a^2 - x^2)^{\frac{7}{2}}}$.

16. Rationalization of expressions containing $\sqrt{x^2 - a^2}$.

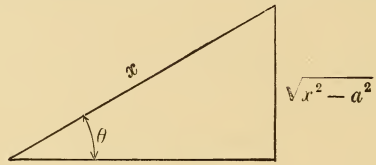


Fig. 3.

From the triangle in Fig. 3,

$$\sqrt{x^2 - a^2} = a \tan\theta,$$

$$x = a \sec\theta,$$

and

$$dx = a \sec\theta \tan\theta\, d\theta.$$

EXAMPLE 1. To integrate $dy = \dfrac{dx}{x\sqrt{x^2 - a^2}}$.

$$dy = \frac{a \sec\theta \tan\theta\, d\theta}{a \sec\theta\, a \tan\theta} = \frac{d\theta}{a},$$

$$y = \frac{\theta}{a} + C = \frac{1}{a}\sec^{-1}\frac{x}{a} + C.$$

Ex. 2. To integrate $dy = \dfrac{\sqrt{x^2 - a^2}}{x} dx$.

$$dy = \frac{a \tan \theta \; a \sec \theta \tan \theta \, d\theta}{a \sec \theta}$$
$$= a \tan^2 \theta \, d\theta = a (\sec^2 \theta - 1) \, d\theta,$$
$$y = a \tan \theta - a\theta + C$$
$$= \sqrt{x^2 - a^2} - a \sec^{-1} \frac{x}{a} + C.$$

Note. The other angle of the triangle could also have been used, in which case, calling this angle α,

$$\sqrt{a^2 + x^2} = a \csc \alpha, \quad x = a \cot \alpha, \text{ and } dx = -a \csc^2 \alpha \, d\alpha.$$
$$\sqrt{a^2 - x^2} = a \sin \alpha, \quad x = a \cos \alpha, \text{ and } dx = -a \sin \alpha \, d\alpha.$$
$$\sqrt{x^2 - a^2} = a \cot \alpha, \quad x = a \csc \alpha, \text{ and } dx = -a \cot \alpha \csc \alpha \, d\alpha.$$

EXAMPLES. X.

1. $dy = \dfrac{\sqrt{x^2 - a^2}\, dx}{x^3}$.

2. $dy = \dfrac{(x^2 - a^2)^{\frac{3}{2}}\, dx}{x^5}$.

3. $dy = \dfrac{dx}{x^2 \sqrt{x^2 - a^2}}$.

4. $dy = \dfrac{(x^2 - a^2)^{\frac{3}{2}}\, dx}{x}$.

5. $dy = \dfrac{\sqrt{x^2 - a^2}\, dx}{x^7}$.

6. $dy = \dfrac{x^3\, dx}{\sqrt{x^2 - a^2}}$.

7. $dy = \dfrac{dx}{x^4 \sqrt{x^2 - a^2}}$.

8. $dy = \dfrac{\sqrt{x^2 - a^2}\, dx}{x^5}$.

17. Change of form of radical. In some cases rationalizing a differential expression by trigonometric substitution as just shown, leads to a trigonometric differential expression which would be integrated by means of an imaginary trigonometric substitution, which, of course, can be done.

On the other hand, since the form of the radical determines the resulting trigonometric function, it is possible, by using the imaginary, to change the form of the radical so that the resulting trigonometric differential expression may at once be integrated.

The sign of both terms under the radical may be changed together by factoring -1 from under the radical, thus:

$$\sqrt{a^2+x^2}=\sqrt{-1\,(-a^2-x^2)}=i\sqrt{-a^2-x^2},$$
$$\sqrt{a^2-x^2}=\sqrt{-1\,(x^2-a^2)}=i\sqrt{x^2-a^2},$$
$$\sqrt{x^2-a^2}=\sqrt{-1\,(a^2-x^2)}=i\sqrt{a^2-x^2}.$$

Also by putting $x=iv$, the sign of x^2 can be changed alone thus:

$$\sqrt{a^2+x^2}=\sqrt{a^2-v^2},$$
$$\sqrt{a^2-x^2}=\sqrt{a^2+v^2},$$
$$\sqrt{x^2-a^2}=\sqrt{-v^2-a^2}.$$

Hence it is possible to change the form of the given radical into any one of the three forms, as follows:

$$\sqrt{a^2+x^2}=\sqrt{a^2-v^2}=i\sqrt{v^2-a^2},$$
$$\sqrt{a^2-x^2}=i\sqrt{x^2-a^2}=\sqrt{a^2+v^2},$$
$$\sqrt{x^2-a^2}=i\sqrt{a^2-x^2}=i\sqrt{a^2+v^2},$$

the form chosen being the one which renders the simplest expression to integrate.

It will now be shown how this applies to a number of examples.

EXAMPLE 1. To integrate $dy = \dfrac{dx}{\sqrt{a^2+x^2}}$.

This expression, if rationalized as shown in Art. 14, would reduce to

$$dy = a \sec \theta \, d\theta$$

and integrated same as Ex. 1, Art. 7, but by changing the form of the radical to $\sqrt{a^2-v^2}$ by putting $x = iv$ and $dx = i\,dv$, a much simpler expression to integrate results, thus:

$$dy = \frac{dx}{\sqrt{a^2+x^2}} = \frac{i\,dv}{\sqrt{a^2-v^2}}$$

$$= \frac{i\,a \cos\theta\,d\theta}{a\cos\theta} = i\,d\theta, \quad \text{(See Fig. 4.)}$$

$$y = i\,\theta + C = \log(\cos\theta + i\sin\theta) + C$$

$$= \log\left(\frac{\sqrt{a^2-v^2}+iv}{a}\right) + C$$

$$= \log\left(\frac{\sqrt{a^2+x^2}+x}{a}\right) + C.$$

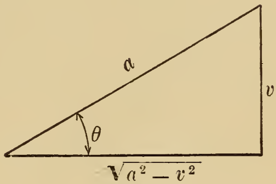


Fig. 4.

$*dy = \dfrac{i\,dv}{\sqrt{a^2-v^2}}$ may also be integrated as $y = i\sin^{-1}\dfrac{v}{a}$,

then $\dfrac{v}{a} = \sin\dfrac{y}{i} = \dfrac{\varepsilon^y - \varepsilon^{-y}}{2i}$ . . . . (23, p. 29.)

$$\frac{2iv}{a} = \frac{2x}{a} = \varepsilon^y - \varepsilon^{-y}$$

multiplying through by ε^y,

$$\varepsilon^{2y} - \frac{2x}{a}\varepsilon^y = 1,$$

* Price, Vol. II, Art. 36.

from which $$\varepsilon^y = \frac{x+\sqrt{a^2+x^2}}{a},$$

and $$y = \log\left(\frac{x+\sqrt{a^2+x^2}}{a}\right) + C.$$

Ex. 2. To integrate $dy = \dfrac{dx}{a^2-x^2}$.

Let $x = iv$, then $dx = i\ dv$,

and $$dy = \frac{i\ dv}{a^2+v^2} = \frac{i\ a \sec^2\theta\ d\theta}{a^2\sec^2\theta} = \frac{i\ d\theta}{a} \quad \text{(See Fig. 5.)}$$

$$y = \frac{i\ \theta}{a} + C = \frac{1}{a}\log\ (\cos\ \theta + i\ \sin\ \theta) + C$$

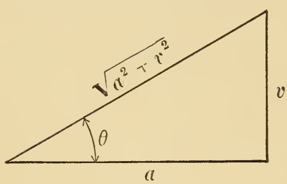


Fig. 5.

$$= \frac{1}{a}\log\left(\frac{a+iv}{\sqrt{a^2+v^2}}\right) + C$$

$$= \frac{1}{a}\log\left(\frac{a+x}{\sqrt{a^2-x^2}}\right) + C$$

$$= \frac{1}{a}\log\sqrt{\frac{a+x}{a-x}} + C.$$

$dy = \dfrac{i\ dv}{a^2+v^2}$ can also be integrated as $y = \dfrac{i}{a}\tan^{-1}\dfrac{v}{a}$,

then $$\frac{v}{a} = \tan\frac{ay}{i} = \frac{\varepsilon^{2ay}-1}{i\ (\varepsilon^{2ay}+1)} \quad . \quad . \quad . \quad . \quad . \quad (24,\ \text{p. } 29.)$$

$$\frac{iv}{a} = \frac{x}{a} = \frac{\varepsilon^{2ay}-1}{\varepsilon^{2ay}+1}.$$

By composition and division in proportion,

$$\frac{a+x}{a-x} = \frac{2\varepsilon^{2ay}}{2} = \varepsilon^{2ay}$$

$$2ay = \log\frac{a+x}{a-x}$$

and $$y = \frac{1}{a}\log\sqrt{\frac{a+x}{a-x}} + C.$$

Ex. 3. To integrate $dy = \frac{\sqrt{a^2 - x^2}}{x} dx$.

$$dy = \frac{i\sqrt{x^2 - a^2}}{x} dx$$

$$= \frac{i\ a \tan\ \theta\ a \sec\ \theta \tan\ \theta\ d\theta}{a \sec\ \theta} \qquad \text{(See Fig. 3.)}$$

$$= ia \tan^2\ \theta\ d\theta = ia\ (\sec^2\ \theta - 1)\ d\theta,$$

$$y = ia \tan\ \theta - ia\theta\ + C$$

$$= ia \tan\ \theta + a \log\ (\cos\ \theta - i \sin\ \theta) + C$$

$$= i\sqrt{x^2 - a^2} + a \log\left(\frac{a - i\sqrt{x^2 - a^2}}{x}\right) + C$$

$$= \sqrt{a^2 - x^2}\ + a \log\left(\frac{a - \sqrt{a^2 - x^2}}{x}\right) + C.$$

Ex. 4. To integrate $dy = \frac{dx}{\sqrt{x^2 - a^2}}$.

$$dy = \frac{dx}{i\sqrt{a^2 - x^2}}$$

$$= -\frac{a \sin\ \theta\ d\theta}{ia \sin\ \theta} = -\frac{d\theta}{i}, \qquad \text{(See Fig. 6.)}$$

$$y = -\frac{\theta}{i} + C = i\theta + C$$

$$= \log\ (\cos\ \theta + i \sin\ \theta) + C$$

$$= \log\left(\frac{x + i\sqrt{a^2 - x^2}}{a}\right) + C$$

$$= \log\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right) + C.$$

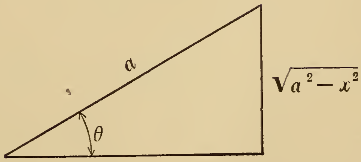


Fig. 6.

Ex. 5. To integrate $dy = \dfrac{dx}{x\sqrt{a^2+x^2}}$.

Let $\qquad x = iv$, then $dx = i\,dv$,

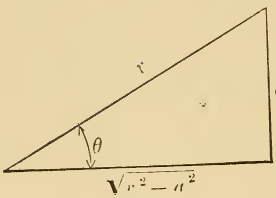


Fig. 7.

and
$$dy = \frac{i\,dv}{iv\sqrt{a^2-v^2}} = \frac{dv}{iv\sqrt{v^2-a^2}}$$
$$= -\frac{a \csc\theta \cot\theta\, d\theta}{ia \csc\theta\; a \cot\theta} \quad \text{(See Fig. 7.)}$$
$$= -\frac{d\theta}{ia} = \frac{i\,d\theta}{a},$$

$$y = \frac{i\theta}{a} + C = \frac{1}{a}\log(\cos\theta + i\sin\theta) + C$$
$$= \frac{1}{a}\log\left(\frac{\sqrt{v^2-a^2}}{v} + \frac{ia}{v}\right) + C$$
$$= \frac{1}{a}\log\left(\frac{i\sqrt{-x^2-a^2}}{x} - \frac{a}{x}\right) + C$$
$$= \frac{1}{a}\log\left(\frac{\sqrt{a^2+x^2}-a}{x}\right) + C.$$

Ex. 6. To integrate $dy = \dfrac{dx}{(x^2-a^2)^{\frac{3}{2}}}$.

Let $\qquad x = iv$, then $dx = i\,dv$,

and
$$dy = \frac{i\,dv}{(-v^2-a^2)^{\frac{3}{2}}} = \frac{i\,dv}{-i(v^2+a^2)^{\frac{3}{2}}} = -\frac{dv}{(a^2+v^2)^{\frac{3}{2}}}$$
$$= -\frac{a\sec^2\theta\, d\theta}{a^3\sec^3\theta} = -\frac{\cos\theta\, d\theta}{a^2}, \quad \text{(See Fig 5.)}$$
$$y = -\frac{\sin\theta}{a^2} + C$$
$$= -\frac{v}{a^2\sqrt{a^2+v^2}} + C$$
$$= -\frac{x}{a^2 i\sqrt{a^2-x^2}} + C$$
$$= -\frac{x}{a^2\sqrt{x^2-a^2}} + C.$$

EXAMPLES. XI.

1. $ay = \sqrt{a^2 + x^2}\, dx.$

2. $dy = \dfrac{dx}{(a^2 - x^2)^2}.$

3. $dy = \dfrac{x^2\, dx}{\sqrt{x^2 - a^2}}.$

4. $dy = \dfrac{dx}{x\,(a^2 - x^2)^{\frac{3}{2}}}.$

5. $dy = \dfrac{dx}{x^2(a^2 - x^2)^2}.$

6. $dy = (1 + x^2)^{\frac{3}{2}}\, dx.$

7. $dy = \dfrac{\sqrt{1 + x^2}\, dx}{x^3}.$

8. $dy = x^2\sqrt{x^2 - a^2}\, dx.$

9. $dy = \dfrac{dx}{x^2 - a^2}.$

10. $dy = \dfrac{dx}{(a^2 - x^2)^3}.$

11. $dy = \dfrac{dx}{x\sqrt{a^2 - x^2}}.$

12. $dy = \dfrac{x^2\, dx}{(x^2 - a^2)^2}.$

13. $dy = \dfrac{\sqrt{a^2 + x^2}\, dx}{x^5}.$

14. $dy = x^2\sqrt{1 + x^2}\, dx.$

15. $dy = \dfrac{dx}{x^3\sqrt{a^2 - x^2}}.$

16. $dy = \dfrac{x^2\, dx}{(a^2 - x^2)^3}.$

17. $dy = \dfrac{\sqrt{a^2 - x^2}\, dx}{x^5}.$

18. $dy = \dfrac{dx}{(x^2 - a^2)^2}.$

19. $dy = \sqrt{x^2 - a^2}\, dx.$

20. $dy = \dfrac{x^2\, dx}{\sqrt{a^2 + x^2}}.$

MISCELLANEOUS EXAMPLES. II.

1. $dy = \dfrac{x^6\,dx}{(a^2+x^2)^4}$.

2. $dy = \dfrac{(x^2-a^2)^{\frac{5}{2}}\,dx}{x}$.

3. $dy = \dfrac{dx}{x\,(x^2-a^2)^{\frac{3}{2}}}$.

4. $dy = x^4\sqrt{a^2-x^2}\,dx$.

5. $dy = \dfrac{dx}{x^2(a^2-x^2)^{\frac{3}{2}}}$.

6. $dy = \dfrac{dx}{x^4(a^2-x^2)}$.

7. $dy = \dfrac{dx}{x^2(a^2+x^2)}$.

8. $dy = \dfrac{dx}{(a^2+x^2)^3}$.

9. $dy = (x^2-a^2)^{\frac{3}{2}}\,dx$.

10. $dy = x^4\sqrt{x^2-a^2}\,dx$.

11. $dy = \dfrac{dx}{(x^2-a^2)^3}$.

12. $dy = x^4\sqrt{a^2+x^2}\,dx$.

13. $dy = x^2(x^2-a^2)^{\frac{3}{2}}\,dx$.

14. $dy = \dfrac{dx}{x^4(a^2-x^2)^{\frac{3}{2}}}$.

15. $dy = \dfrac{(a^2-x^2)^{\frac{3}{2}}\,dx}{x^8}$.

16. $dy = \dfrac{dx}{(a^2-x^2)^4}$.

17. $dy = \dfrac{dx}{x^4\sqrt{a^2-x^2}}$.

18. $dy = \dfrac{x^5\,dx}{(a^2-x^2)^2}$.

19. $dy = \dfrac{x^7\,dx}{(a^2-x^2)^4}$.

20. $dy = \dfrac{x^4\,dx}{\sqrt{a^2+x^2}}$.

21. $dy = \dfrac{x^3+x^2+a^2}{(a^2+x^2)^2}\,dx$.

22. $dy = \dfrac{dx}{x(a+bx^2)^2}$.

23. $dy = \dfrac{dx}{(a^2 - x^2)\sqrt{b^2 - x^2}}.$

Let $\quad x = a \sin\theta$, then $dx = a\cos\theta\, d\theta$ and

$$dy = \frac{d\theta}{a\cos\theta\sqrt{b^2 - a^2\sin^2\theta}} = \frac{\sec^2\theta\, d\theta}{a\sqrt{b^2\sec^2\theta - a^2\tan^2\theta}}.$$

When $a > b$,

$$dy = \frac{d(\tan\theta)}{a\sqrt{b^2 - (a^2 - b^2)\tan^2\theta}}$$

$$y = \frac{1}{a\sqrt{a^2 - b^2}}\sin^{-1}\left(\frac{\sqrt{a^2 - b^2}\tan\theta}{b}\right) + C$$

$$= \frac{1}{a\sqrt{a^2 - b^2}}\sin^{-1}\left(\frac{x\sqrt{a^2 - b^2}}{b\sqrt{a^2 - x^2}}\right) + C.$$

When $a < b$,

$$dy = \frac{d(\tan\theta)}{a\sqrt{b^2 + (b^2 - a^2)\tan^2\theta}}$$

$$y = \frac{1}{a\sqrt{b^2 - a^2}}\log\left(\frac{\sqrt{b^2 + (b^2 - a^2)\tan^2\theta} + \sqrt{b^2 - a^2}\tan\theta}{b}\right) + C$$

$$= \frac{1}{a\sqrt{b^2 - a^2}}\log\left(\frac{a\sqrt{b^2 - x^2} + x\sqrt{b^2 - a^2}}{b\sqrt{a^2 - x^2}}\right) + C.$$

18. Rationalization of expression containing $\sqrt{2ax - x^2}$.

This surd may be rationalized in two ways, either as $\sqrt{x}\,\sqrt{2a - x}$ or as $\sqrt{a^2 - (x - a)^2}$.

In the first case, from Fig. 8,

$$\sqrt{2a - x} = \sqrt{2a}\cos\theta,$$
$$x = 2a\sin^2\theta,$$
$$\sqrt{2ax - x^2} = 2a\sin\theta\cos\theta,$$

and $\quad dx = 4a\sin\theta\cos\theta\, d\theta.$

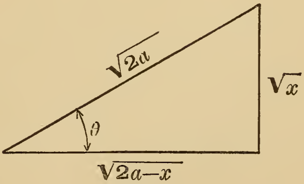


Fig. 8.

In the second case, from Fig. 9,

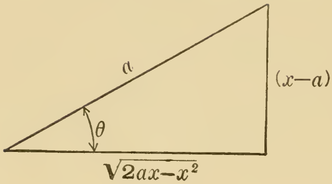


Fig. 9.

$$\sqrt{a^2-(x-a)^2}=a\cos\theta,$$
$$x-a=a\sin\theta,$$
$$x=a\,(1+\sin\theta),$$
$$dx=a\cos\theta\,d\theta.$$

The latter case is the simpler one, and can always be used except when the radical appears in the denominator, multiplied by some power of x, when the first case is used.

EXAMPLE 1. To integrate $dy=\dfrac{dx}{\sqrt{2ax-x^2}}$.

$$dy=\frac{4a\sin\theta\cos\theta\,d\theta}{2a\sin\theta\cos\theta}=2d\theta,\qquad\text{(See Fig. 8.)}$$
$$y=2\,\theta+C$$
$$=2\sin^{-1}\sqrt{\frac{x}{2a}}+C,$$

or

$$dy=\frac{a\cos\theta\,d\theta}{a\cos\theta}=d\theta,\qquad\text{(See Fig. 9.)}$$
$$y=\theta+C$$
$$=\sin^{-1}\left(\frac{x}{a}-1\right)+C.$$

19. Rationalization of expressions containing $\sqrt{2ax+x^2}$. This surd may be rationalized, either as $\sqrt{x}\,\sqrt{2a+x}$ or $\sqrt{(x+a)^2-a^2}$.

From Fig. 10,

$$\sqrt{2a+x}=\sqrt{2a}\sec\theta,$$
$$x=2a\tan^2\theta,$$
$$\sqrt{2ax+x^2}=2a\sec\theta\tan\theta,$$

and
$$dx=4a\tan\theta\sec^2\theta\,d\theta.$$

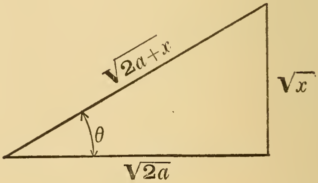


Fig. 10.

From Fig. 11,

$$\sqrt{2ax+x^2}=a\tan\theta,$$

$$x+a=a\sec\theta,$$

$$x=a(\sec\theta-1),$$

and $$dx=a\sec\theta\tan\theta\,d\theta.$$

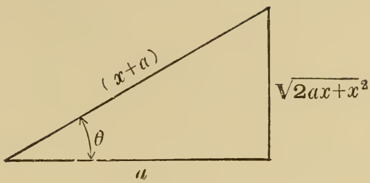


Fig. 11.

EXAMPLE 1. To integrate $dy=\dfrac{dx}{(2ax+x^2)^{\frac{3}{2}}}$.

$$dy=\frac{4a\tan\theta\sec^2\theta\,d\theta}{8a^3\sec^3\theta\tan^3\theta}\qquad\text{(See Fig. 10.)}$$

$$=\frac{d\theta}{2a^2\sec\theta\tan^2\theta}$$

$$=\frac{\cos^3\theta\,d\theta}{2a^2\sin^2\theta}=\frac{1}{2a^2}\left(\frac{1-\sin^2\theta}{\sin^2\theta}\right)d(\sin\theta),$$

$$y=\frac{1}{2a^2}(-\csc\theta-\sin\theta)+C$$

$$=-\frac{1}{2a^2}\left(\sqrt{\frac{2a+x}{x}}+\sqrt{\frac{x}{2a+x}}\right)+C$$

$$=-\frac{2(a+x)}{2a^2\sqrt{2ax+x^2}}+C=-\frac{a+x}{a^2\sqrt{2ax+x^2}}+C;$$

or, $$dy=\frac{a\sec\theta\tan\theta\,d\theta}{a^3\tan^3\theta}\qquad\text{(See Fig. 11.)}$$

$$=\frac{1}{a^2}\left(\frac{\cos\theta}{\sin^2\theta}\right)d\theta=\frac{1}{a^2}\cot\theta\csc\theta\,d\theta,$$

$$y=-\frac{1}{a^2}\csc\theta+C$$

$$=-\frac{x+a}{a^2\sqrt{2ax+x^2}}+C$$

EXAMPLES. XII.

1. $dy = \dfrac{x^2\,dx}{\sqrt{2ax - x^2}}$.

2. $dy = \dfrac{x^3\,dx}{\sqrt{2ax - x^2}}$.

3. $dy = \dfrac{x\,dx}{\sqrt{2ax - x^2}}$.

4. $dy = \sqrt{2ax - x^2}\,dx$.

5. $dy = x\sqrt{2ax - x^2}\,dx$.

6. $dy = \dfrac{dx}{x\sqrt{2ax - x^2}}$

$$= \frac{d\theta}{a\,(1 + \sin\theta)} = \frac{1 - \sin\theta}{\cos^2\theta}\,d\theta.$$

(See Fig. 9.)

7. $dy = \sqrt{2ax + x^2}\;dx$

$$= i\sqrt{a^2 - (x + a)^2}\;dx.$$

Let $(x + a) = a\cos\theta$.

8. $dy = \dfrac{x\,dx}{\sqrt{ax - x^2}}$.

20. Rationalization of expressions containing trinomial surds.

Differential expressions, not too complicated, containing trinomial surds may be rationalized and integrated on a plan similar to the one followed in the two preceding articles.

Example 1. To integrate $dy = \dfrac{dx}{(x^2 + 2x + 10)^{\frac{3}{2}}}$.

$$dy = \frac{dx}{[\,(x+1)^2 + 9]^{\frac{3}{2}}}$$

$$= \frac{3\sec^2\theta\;d\theta}{27\sec^3\theta} \qquad \text{(See Fig. 12.)}$$

$$= \frac{\cos\theta\,d\theta}{9},$$

$$y = \frac{\sin\theta}{9} + C$$

$$= \frac{x + 1}{9\sqrt{x^2 + 2x + 10}} + C.$$

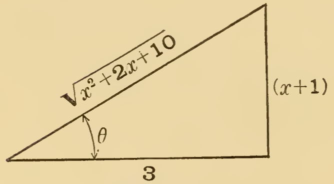


Fig. 12.

21. Binomial Differentials. Every binomial differential expression may be reduced to the form

$$x^m (a + bx^n)^{\frac{p}{q}} \, dx,$$

where m, n, p, q are integers and n is positive.

As found in text-books this binomial differential expression may be rationalized and integrated in the following cases:

Case I. When $\dfrac{m+1}{n}$ = an integer or zero, by assuming

$$a + bx^n = z^q.$$

Case II. When $\dfrac{m+1}{n} + \dfrac{p}{q}$ = an integer or zero, by assuming

$$a + bx^n = z^q x^n.$$

The student should now be able to prove that this binomial differential expression can be rationalized by trigonometric substitution in the cases just given, and therefore it is not necessary for him to remember these conditions, since he can readily determine by trial whether a given binomial differential expression can be rationalized by trigonometric substitution.

EXAMPLE 1. To integrate $dy = \dfrac{dx}{x^2 (a^2 + x^3)^{\frac{5}{3}}}$.

Let $\sqrt{a^2 + x^3} = a \sec \theta,$

then $x = a^{\frac{2}{3}} \tan^{\frac{2}{3}} \theta,$

and $dx = \frac{2}{3} a^{\frac{2}{3}} \tan^{-\frac{1}{3}} \theta \sec^2 \theta \, d\theta,$ (Draw the triangle.)

$$dy = \frac{2 \, d\theta}{3a^4 \tan^{\frac{5}{3}} \theta \sec^{\frac{4}{3}} \theta}$$

$$= \frac{2 \cos^3 \theta \, d\theta}{3a^4 \sin^{\frac{5}{3}} \theta},$$

$$y = \frac{2}{3a^4}\left(-\frac{3}{2}\sin^{-\frac{2}{3}}\theta - \frac{3\sin^{\frac{4}{3}}\theta}{4}\right) + C$$

$$= -\frac{(a^2+x^3)^{\frac{1}{3}}}{a^4\,x} - \frac{x^2}{2a^4\,(a^2+x^3)^{\frac{2}{3}}} + C$$

$$= -\frac{2a^2+3x^3}{2a^4\,x\,(a^2+x^3)^{\frac{2}{3}}} + C.$$

EXAMPLES. XIII.

1. $dy = \dfrac{dx}{\sqrt{2+3x+x^2}}$.

2. $dy = \dfrac{dx}{\sqrt{2+x-x^2}}$.

3. $dy = \sqrt{2+x-x^2}\,dx$.

4. $dy = \dfrac{x^2\,dx}{(a^2+x^3)^{\frac{5}{3}}}$.

5. $dy = x^5\,(1+x^2)^{\frac{2}{3}}\,dx$.

6. $dy = \dfrac{(3x+2)\,dx}{(x^2-3x+3)^2}$.

7. $dy = \dfrac{dx}{x\,(a+bx^n)^2}$.

8. $dy = x^3(a^2-x^2)^{\frac{1}{2}}\,dx$.

ANSWERS TO EXAMPLES.

EXAMPLES. I. PAGE 31.

1. $y = \frac{\sin^2 x}{2} + C$, or $y = -\frac{\cos^2 x}{2} + C$.

2. $y = \frac{\cos^3 x}{3} - \cos x + C$.

3. $y = -\log \cos x + C$, or $y = \log \sec x + C$.

4. $y = \log \sin x + C$, or $y = -\log \csc x + C$.

5. $y = \frac{\sin^3 x}{3} - \frac{2}{5} \sin^5 x + \frac{\sin^7 x}{7} + C$.

6. $y = \log \csc x - \frac{\csc^2 x}{2} + C$.

7. $y = \frac{\cos^7 x}{7} - \frac{3}{5} \cos^5 x + \cos^3 x - \cos x + C$.

8. $y = \sin x - \frac{\sin^3 x}{3} + C$.

9. $y = \frac{\cos^2 x}{2} - \log \cos x + C$.

10. $y = \sin x - \sin^3 x + \frac{3}{5} \sin^5 x - \frac{\sin^7 x}{7} + C$.

11. $y = \log \csc x - \frac{3}{2} \csc^2 x + \frac{3}{4} \csc^4 x - \frac{\csc^6 x}{6} + C$.

12. $y = \frac{\sec^4 x}{4} - \sec^2 x + \log \sec x + C$.

13. $y = \frac{\sec^3 x}{3} - \sec x + C$.

14. $y = \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$.

15. $y = \frac{\sec^2 x}{2} + 2 \log \cos x - \frac{\cos^2 x}{2} + C$.

16. $y = -\csc x - \sin x + C$.

EXAMPLES. II. PAGE 32.

1. $y = \tan x + \dfrac{\tan^3 x}{3} + C.$

2. $y = \tan x + \dfrac{2}{3} \tan^3 x + \dfrac{\tan^5 x}{5} + C.$

3. $y = -\cot x - \dfrac{\cot^3 x}{3} + C.$

4. $y = -\cot x - \dfrac{2}{3} \cot^3 x - \dfrac{\cot^5 x}{5} + C.$

5. $y = \log \tan x + C.$

6. $y = -\cot x + \tan x + C.$

7. $y = \dfrac{\tan^3 x}{3} + C.$

8. $y = -\dfrac{\cot^3 x}{3} - \dfrac{2}{5} \cot^5 x - \dfrac{\cot^7 x}{7} + C.$

9. $y = \dfrac{\tan^4 x}{4} + \dfrac{\tan^6 x}{6} + C.$

10. $y = -\cot x + 2 \tan x + \dfrac{\tan^3 x}{3} + C.$

EXAMPLES. III. PAGE 33.

1. $y = \dfrac{x}{2} + \dfrac{\sin 2x}{4} + C.$

2. $y = \dfrac{3x}{8} - \dfrac{\sin 2x}{4} + \dfrac{\sin 4x}{32} + C.$

3. $y = \dfrac{3x}{8} + \dfrac{\sin 2x}{4} + \dfrac{\sin 4x}{32} + C.$

4. $y = \dfrac{x}{16} - \dfrac{\sin 4x}{64} - \dfrac{\sin^3 2x}{48} + C.$

5. $y = \dfrac{3x}{128} - \dfrac{\sin 4x}{128} + \dfrac{\sin 8x}{1024} + C.$

EXAMPLES. IV. PAGE 39.

1. $y = \frac{1}{2}\log(\sec x + \tan x) + \frac{\tan x \sec x}{2} + C.$

2. $y = \frac{3}{8}\log(\sec x + \tan x) + \frac{5}{8}\tan x \sec x$
$+ \frac{\tan^3 x \sec x}{4} + C.$

3. $y = \frac{1}{2}\log(\csc x - \cot x) - \frac{\cot x \csc x}{2} + C.$

4. $y = \frac{3}{8}\log(\csc x - \cot x) - \frac{5}{8}\cot x \csc x$
$- \frac{\cot^3 x \csc x}{4} + C.$

5. $y = \frac{1}{2}\log(\sec x - \tan x) + \frac{\tan x \sec x}{2} + C.$

6. $y = \frac{3}{8}\log(\sec x + \tan x) - \frac{3}{8}\tan x \sec x$
$+ \frac{\tan^3 x \sec x}{4} + C.$

7. $y = \frac{1}{8}\log(\csc x + \cot x) - \frac{\cot x \csc x}{8} - \frac{\cot^3 x \csc x}{4} + C.$

8. $y = \frac{3}{8}\log(\csc x - \cot x) + \frac{3}{8}\cot x \csc x$
$- \frac{\cot^3 x \csc x}{4} + C.$

9. $y = \frac{3}{2}\log(\sec x + \tan x) + \frac{\tan x \sec x}{2} - \csc x + C.$

10. $y = \sec x + \log(\csc x - \cot x) + C.$

11. $y = \log(\sec x + \tan x) - \csc x - \frac{\csc^3 x}{3} + C.$

12. $y = \frac{1}{16}\log(\sec x + \tan x) - \frac{\tan x \sec x}{16} - \frac{\tan^3 x \sec x}{8}$
$+ \frac{\tan^3 x \sec^3 x}{6} + C.$

EXAMPLES. V. PAGE 40.

1. $y = \tan x - x + C.$

2. $y = \log(\sec x + \tan x) - \sin x + C.$

3. $y = \dfrac{\tan^3 x}{3} - \tan x + x + C.$

4. $y = -\cot x - x + C.$

5. $y = -\dfrac{\cot^5 x}{5} + \dfrac{\cot^3 x}{3} - \cot x - x + C.$

6. $y = \dfrac{3}{2}\log(\csc x + \cot x) - \dfrac{\cot x \csc x}{2} - \cos x + C.$

7. $y = \dfrac{\tan x \sec x}{2} + \dfrac{3}{2}\log(\sec x - \tan x) + \sin x + C.$

EXAMPLES. VI. PAGE 41.

1. $y = -\dfrac{\cos 8x}{16} - \dfrac{\cos 2x}{4} + C.$

2. $y = -\dfrac{\sin 11x}{22} + \dfrac{\sin x}{2} + C.$

3. $y = \dfrac{\sin 11x}{22} + \dfrac{\sin 3x}{6} + C.$

4. $y = -\dfrac{\cos 2x}{4} + \dfrac{3}{4}\cos\dfrac{2}{3}x + C.$

5. $y = -\dfrac{\cos x}{2} + \cos\dfrac{1}{2}x + C.$

6. $y = \dfrac{3}{26}\sin\dfrac{13}{3}x + \dfrac{3}{10}\sin\dfrac{5}{3}x + C.$

EXAMPLES VII. PAGE 45.

1. $y = \dfrac{\varepsilon^x}{\sqrt{2}}\left(\cos x - \dfrac{\pi}{4}\right) + C.$

2. $y = \dfrac{\varepsilon^x}{\sqrt{2}}\sin\left(x - \dfrac{\pi}{4}\right) + C.$

3. $y = \dfrac{\varepsilon^{2x}}{\sqrt{13}}\left(\cos 3x - \tan^{-1}\dfrac{3}{2}\right) + C.$

4. $y = -\dfrac{1}{\sqrt{2}\,\varepsilon^x}\sin\left(x + \dfrac{\pi}{4}\right) + C.$

5. $y = -\dfrac{\varepsilon^{-3x}}{\sqrt{13}}\sin\left(2x - \tan^{-1}\dfrac{3}{2}\right) + C.$

6. $I = \dfrac{e_m}{\sqrt{R^2 + L^2\omega^2}}\sin\left(\omega t - \tan^{-1}\dfrac{L\omega}{R}\right) + k\varepsilon^{-\frac{R}{L}t}.$

7. $I = \dfrac{e_m}{\sqrt{R^2 + \dfrac{1}{C^2\,\omega^2}}}\sin\left(\omega t + \tan^{-1}\dfrac{1}{RC\omega}\right) + k\varepsilon^{-\frac{t}{RC}}.$

MISCELLANEOUS EXAMPLES. I. PAGE 46.

1. $y = -\cos x + \dfrac{2}{3}\cos^3 x - \dfrac{\cos^5 x}{5} + C.$

2. $y = \dfrac{5x}{16} - \dfrac{\sin 2x}{4} + \dfrac{\sin^3 2x}{48} + \dfrac{3\sin 4x}{64} + C.$

3. $y = \dfrac{\cos^7 x}{7} - \dfrac{\cos^5 x}{5} + C.$

4. $y = \dfrac{\sec^3 x}{3} + C.$

5. $y = \dfrac{\tan^5 x}{5} - \dfrac{\tan^3 x}{3} + \tan x - x + C.$

6. $y = \dfrac{\sec^6 x}{6} - \dfrac{3\sec^4 x}{4} + \dfrac{3\sec^2 x}{2} - \log\sec x + C.$

7. $y = -\dfrac{\cot^3 x}{3} + \cot x + x + C.$

8. $y = -\dfrac{\csc^4 x}{4} + \csc^2 x - \log\csc x + C.$

9. $y = \frac{5}{16} \log(\sec x + \tan x) + \frac{11}{16} \tan x \sec x$
$$+ \frac{3}{8} \tan^3 x \sec x + \frac{\tan^3 x \sec^3 x}{6} + C.$$

10. $y = \tan x + \tan^3 x + \frac{3}{5} \tan^5 x + \frac{\tan^7 x}{7} + C.$

11. $y = \sin x - \frac{2}{3} \sin^3 x + \frac{\sin^5 x}{5} + C.$

12. $y = \frac{5x}{16} + \frac{\sin 2x}{4} - \frac{\sin^3 2x}{48} + \frac{3 \sin 4x}{64} + C.$

13. $y = \frac{5}{16} \log(\csc x - \cot x) - \frac{11}{16} \cot x \csc x$
$$- \frac{3}{8} \cot^3 x \csc x - \frac{\cot^3 x \csc^3 x}{6} + C.$$

14. $y = -\cot x - \cot^3 x - \frac{3}{5} \cot^5 x - \frac{\cot^7 x}{7} + C.$

15. $y = \frac{1}{16} \log(\sec x - \tan x) + \frac{\tan x \sec x}{16} + \frac{\tan^3 x \sec x}{8}$
$$+ \frac{\tan^3 x \sec^3 x}{6} + C.$$

16. $y = \frac{5x}{128} - \frac{\sin^3 2x}{48} - \frac{\sin 4x}{128} - \frac{\sin 8x}{1024} + C.$

17. $y = \tan x - 2 \cot x - \frac{\cot^3 x}{3} + C.$

18. $y = -\frac{\cot^5 x}{5} - \frac{\cot^7 x}{7} + C.$

19. $y = \frac{\sec^3 x}{3} + \sec x + \log(\csc x - \cot x) + C.$

20. $y = -2\sqrt{\cos x}\left(1 - \frac{2 \cos^2 x}{5} + \frac{\cos^4 x}{9}\right) + C.$

21. $y = \dfrac{\varepsilon^{2x}}{2\sqrt{5}} \cos(4x - \tan^{-1} 2) + C.$

22. $y = \dfrac{\sin 9x}{18} + \dfrac{\sin x}{2} + C.$

23. $y = \dfrac{\sin^{\frac{5}{3}} x^3}{5} - \dfrac{\sin^{\frac{11}{3}} x^3}{11} + C.$

24. $y = \dfrac{\sec^7 x}{7} - \dfrac{2\sec^5 x}{5} + \dfrac{\sec^3 x}{3} + C.$

25. $y = \log(\sec x + \tan x) - \sin x - \dfrac{\sin^3 x}{3} + C.$

26. $y = \dfrac{1}{16}\log(\csc x - \cot x) + \dfrac{\cot x \csc x}{16} + \dfrac{\cot^3 x \csc x}{8} - \dfrac{\cot^3 x \csc^3 x}{6} + C.$

27. $y = \dfrac{1}{16}\log(\csc x + \cot x) - \dfrac{\cot x \csc x}{16} - \dfrac{\cot^3 x \csc x}{8} - \dfrac{\cot^3 x \csc^3 x}{6} + C.$

28. $y = \dfrac{5x}{128} + \dfrac{\sin^3 2x}{48} - \dfrac{\sin 4x}{128} - \dfrac{\sin 8x}{1024} + C.$

29. $y = \dfrac{15}{8}\log(\csc x - \cot x) + \dfrac{7}{8}\cot x \csc x - \dfrac{\cot^3 x \csc x}{4} + \cos x + C.$

30. $y = 2\sqrt{\sin x}\left(1 - \dfrac{2}{5}\sin^2 x + \dfrac{\sin^4 x}{9}\right) + C.$

31. $y = -\dfrac{\csc^5 x}{5} + C.$

32. $y = \dfrac{\varepsilon^{3x}}{5}\cos\left(4x - \tan^{-1}\dfrac{4}{3}\right) + C.$

EXAMPLES. VIII. PAGE 48.

1. $y = -\dfrac{3x^2 + 2}{3(1 + x^2)^{\frac{3}{2}}} + C.$

2. $y = \dfrac{15a^2\, x + 20a^2\, x^3 + 8x^5}{15a^6\, (a^2 + x^2)^{\frac{5}{2}}} + C.$

3. $y = \dfrac{1}{a^2} \log \dfrac{x}{\sqrt{a^2 + x^2}} + C.$

4. $y = -\dfrac{a^2 + x^2}{2a^4\, x^2} - \dfrac{1}{a^4} \log \dfrac{x}{\sqrt{a^2 + x^2}} + C.$

5. $y = \dfrac{3a^2\, x + 2\, x^3}{3a^4\, (a^2 + x^2)^{\frac{3}{2}}} + C.$

6. $y = \log \dfrac{\sqrt{a^2 + x^2}}{a} + \dfrac{a^2}{2\, (a^2 + x^2)} + C.$

7. $y = a^4 \log \dfrac{\sqrt{a^2 + x^2}}{a} + \dfrac{(a^2 + x^2)\, (x^2 - 3a^2)}{4} + C.$

8. $y = \dfrac{\sqrt{a^2 + x^2}}{3} (x^2 - 2a^2) + C.$

9. $y = \dfrac{(1 + x^2)^{\frac{3}{2}}}{15} (3x^2 - 2) + C.$

10. $y = \dfrac{3}{8a} \tan^{-1} \dfrac{x}{a} - \dfrac{3x}{8\, (a^2 + x^2)} - \dfrac{x^3}{4\, (a^2 + x^2)^2} + C.$

11. $y = \dfrac{4x^2 + 3}{4\, (x^2 + 1)^2} + \log \sqrt{x^2 + 1} + C.$

12. $y = \dfrac{1}{8a^3} \tan^{-1} \dfrac{x}{a} + \dfrac{x^3 - a^2\, x}{8a^2\, (a^2 + x^2)^2} + C.$

EXAMPLES. IX. PAGE 50.

1. $y = \frac{a^2}{2} \sin^{-1} \frac{x}{a} - \frac{x\sqrt{a^2 - x^2}}{2} + C.$

2. $y = \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x\sqrt{a^2 - x^2}}{2} + C.$

3. $y = -\frac{\sqrt{a^2 - x^2}}{15} (8a^4 + 4a^2 x^2 + 3x^4) + C.$

4. $y = \frac{3a^4}{8} \sin^{-1} \frac{x}{a} + \frac{x}{8} \sqrt{a^2 - x^2} (5a^2 - 2x^2) + C.$

5. $y = \frac{x}{\sqrt{a^2 - x^2}} - \sin^{-1} \frac{x}{a} + C.$

6. $y = \frac{a^2}{2 (a^2 - x^2)} + \log \frac{\sqrt{a^2 - x^2}}{a} + C.$

7. $y = -\frac{\sqrt{a^2 - x^2}}{x} - \sin^{-1} \frac{x}{a} + C.$

8. $y = -\frac{(1 - x^2)^{\frac{5}{2}}}{5x^5} + \frac{(1 - x^2)^{\frac{3}{2}}}{3x^3} - \frac{\sqrt{1 - x^2}}{x} - \sin^{-1} x + C.$

9. $y = -\frac{\sqrt{a^2 - x^2}}{a^2 x} + C.$ 10. $y = \frac{x}{a^2 \sqrt{a^2 - x^2}} + C.$

11. $y = \frac{a^4}{8} \sin^{-1} \frac{x}{a} + \frac{x}{8} \sqrt{a^2 - x^2} (2x^2 - a^2) + C.$

12. $y = \frac{1}{16} \sin^{-1} x + \frac{x}{16} \sqrt{1 - x^2} (2x^2 - 1) + \frac{x^3}{6} (1 - x^2)^{\frac{3}{2}} + C.$

13. $y = \frac{1}{a^2} \log \frac{x}{\sqrt{a^2 - x^2}} + C.$ 14. $y = \frac{5x^2 - 2a^2}{15 (a^2 - x^2)^{\frac{5}{2}}} + C.$

EXAMPLES. X. PAGE 51.

1. $y = \dfrac{1}{2a}\sec^{-1}\dfrac{x}{a} - \dfrac{\sqrt{x^2 - a^2}}{2x^2} + C.$

2. $y = \dfrac{3}{8a}\sec^{-1}\dfrac{x}{a} + \dfrac{\sqrt{x^2 - a^2}}{8x^4}(2a^2 - 5x^2) + C.$

3. $y = \dfrac{\sqrt{x^2 - a^2}}{a^2 x} + C.$

4. $y = \left(\dfrac{x^2 - 4a^2}{3}\right)\sqrt{x^2 - a^2} + a^3\sec^{-1}\dfrac{x}{a} + C.$

5. $y = \dfrac{1}{16a^5}\sec^{-1}\dfrac{x}{a} - \dfrac{\sqrt{x^2 - a^2}}{16a^4 x^2} + \dfrac{(x^2 - a^2)^{\frac{3}{2}}}{8a^4 x^4} + \dfrac{(x^2 - a^2)^{\frac{3}{2}}}{6a^2 x^6} + C.$

6. $y = \dfrac{\sqrt{x^2 - a^2}}{3}(x^2 + 2a^2) + C.$

7. $y = \dfrac{\sqrt{x^2 - a^2}}{3a^4 x^3}(2x^2 + a^2) + C.$

8. $y = \dfrac{1}{8a^3}\sec^{-1}\dfrac{x}{a} + \dfrac{\sqrt{x^2 - a^2}}{8a^2 x^4}(x^2 - 2a^2) + C.$

EXAMPLES. XI. PAGE 57.

1. $y = \dfrac{a^2}{2}\log\left(\dfrac{\sqrt{a^2 + x^2} + x}{a}\right) + \dfrac{x}{2}\sqrt{a^2 + x^2} + C.$

2. $y = \dfrac{1}{2a^3}\log\sqrt{\dfrac{a + x}{a - x}} + \dfrac{x}{2a^2(a^2 - x^2)} + C.$

3. $y = \dfrac{a^2}{2}\log\left(\dfrac{x + \sqrt{x^2 - a^2}}{a}\right) + \dfrac{x\sqrt{x^2 - a^2}}{2} + C.$

4. $y = \dfrac{1}{a^2\sqrt{a^2 - x^2}} + \dfrac{1}{a^3} \log \left(\dfrac{a - \sqrt{a^2 - x^2}}{x}\right) + C.$

5. $y = \dfrac{3}{2a^5} \log \sqrt{\dfrac{a+x}{a-x}} + \dfrac{x}{2a^4(a^2 - x^2)} - \dfrac{1}{a^4 x} + C.$

6. $y = \dfrac{3}{8} \log (x + \sqrt{1 + x^2}) + \dfrac{x\sqrt{1 + x^2}}{8} (5 + 2x^2) + C.$

7. $y = \dfrac{1}{2} \log \left(\dfrac{\sqrt{1 + x^2} - 1}{x}\right) - \dfrac{\sqrt{1 + x^2}}{2x^2} + C.$

8. $y = \dfrac{a^4}{8} \log \left(\dfrac{x - \sqrt{x^2 - a^2}}{a}\right) + \dfrac{x\sqrt{x^2 - a^2}}{8} (2x^2 - a^2) + C.$

9. $y = \dfrac{1}{a} \log \sqrt{\dfrac{x-a}{x+a}} + C.$

10. $y = \dfrac{3}{8a^5} \log \sqrt{\dfrac{a+x}{a-x}} + \dfrac{5x}{8a^4(a^2 - x^2)} + \dfrac{x^3}{4a^4(a^2 - x^2)^2} + C.$

11. $y = \dfrac{1}{a} \log \dfrac{a - \sqrt{a^2 - x^2}}{x} + C.$

12. $y = \dfrac{1}{2a} \log \sqrt{\dfrac{x-a}{x+a}} - \dfrac{x}{2(x^2 - a^2)} + C.$

13. $y = \dfrac{1}{8a^3} \log \left(\dfrac{a + \sqrt{a^2 + x^2}}{x}\right) - \dfrac{\sqrt{a^2 + x^2}}{8a^2 x^4} (x^2 + 2a^2) + C.$

14. $y = \dfrac{1}{8} \log (\sqrt{1 + x^2} - x) + \dfrac{x\sqrt{1 + x^2}}{8} (1 + 2x^2) + C.$

15. $y = \dfrac{1}{2a^3} \log \left(\dfrac{a - \sqrt{a^2 - x^2}}{x}\right) - \dfrac{\sqrt{a^2 - x^2}}{2a^2 x^2} + C.$

16. $y = \dfrac{1}{8a^3} \log \sqrt{\dfrac{a-x}{a+x}} + \dfrac{x}{8a^2(a^2 - x^2)^2} (a^2 + x^2) + C.$

17. $y = \dfrac{1}{8a^3} \log \left(\dfrac{a + \sqrt{a^2 - x^2}}{x} \right) - \dfrac{\sqrt{a^2 - x^2}}{8a^2 x^4} (2a^2 - x^2) + C.$

18. $y = \dfrac{1}{2a^3} \log \sqrt{\dfrac{x+a}{x-a}} - \dfrac{x}{2a^2 (x^2 - a^2)} + C.$

19. $y = \dfrac{a^2}{2} \log \left(\dfrac{x - \sqrt{x^2 - a^2}}{a} \right) + \dfrac{x\sqrt{x^2 - a^2}}{2} + C.$

20. $y = \dfrac{a^2}{2} \log \left(\dfrac{\sqrt{a^2 + x^2} - x}{a} \right) + \dfrac{x\sqrt{a^2 + x^2}}{2} + C.$

MISCELLANEOUS EXAMPLES. II. PAGE 58.

1. $y = \dfrac{5}{16a} \tan^{-1} \dfrac{x}{a} - \dfrac{5x}{16 (a^2 + x^2)} - \dfrac{3x^3}{8 (a^2 + x^2)^2}$
$\qquad + \dfrac{a^2 x^3}{6 (a^2 + x^2)^3} + C.$

2. $y = \dfrac{(x^2 - a^2)^{\frac{5}{2}}}{5} - \dfrac{a^2 (x^2 - a^2)^{\frac{3}{2}}}{3} + a^4 \sqrt{x^2 - a^2} - a^5 \sec^{-1} \dfrac{x}{a} + C.$

3. $y = - \dfrac{1}{a^2 \sqrt{x^2 - a^2}} - \dfrac{1}{a^3} \sec^{-1} \dfrac{x}{a} + C.$

4. $y = \dfrac{a^6}{16} \sin^{-1} \dfrac{x}{a} - \dfrac{a^4 x}{16} \sqrt{a^2 - x^2} + \dfrac{a^2 x^3}{8} \sqrt{a^2 - x^2}$
$\qquad - \dfrac{x^3 (a^2 - x^2)^{\frac{3}{2}}}{6} + C.$

5. $y = \dfrac{2x^2 - a^2}{a^4 x \sqrt{a^2 - x^2}} + C.$

6. $y = \frac{1}{a^5} \log \sqrt{\frac{a+x}{a-x}} - \frac{1}{a^4 x} - \frac{1}{3a^2 x^3} + C.$

7. $y = -\frac{1}{a^2 x} - \frac{1}{a^3} \tan^{-1} \frac{x}{a} + C.$

8. $y = \frac{3}{8a^5} \tan^{-1} \frac{x}{a} + \frac{5x}{8a^4 (a^2+x^2)} - \frac{x^3}{4a^4 (a^2+x^2)^2} + C.$

9. $y = \frac{3a^4}{8} \log \left(\frac{x + \sqrt{x^2 - a^2}}{a}\right) - \frac{3a^2 x}{8} \sqrt{x^2 - a^2}$
$+ \frac{x}{4} (x^2 - a^2)^{\frac{3}{2}} + C.$

10. $y = \frac{a^6}{16} \log \left(\frac{x - \sqrt{x^2 - a^2}}{a}\right) + \frac{a^4 x \sqrt{x^2 - a^2}}{16} + \frac{a^2 x}{8} (x^2 - a^2)^{\frac{3}{2}}$
$+ \frac{x^3}{6} (x^2 - a^2)^{\frac{3}{2}} + C.$

11. $y = \frac{3}{8a^5} \log \sqrt{\frac{x-a}{x+a}} + \frac{3x}{8a^4 (x^2 - a^2)} - \frac{x}{4a^2 (x^2 - a^2)^2} + C.$

12. $y = \frac{a^6}{16} \log \left(\frac{x + \sqrt{a^2 + x^2}}{a}\right) - \frac{a^4 x}{16} \sqrt{a^2 + x^2}$
$- \frac{a^2 x^3}{8} \sqrt{a^2 + x^2} + \frac{x^3}{6} (a^2 + x^2)^{\frac{3}{2}}.$

13. $y = \frac{a^6}{16} \log \left(\frac{x + \sqrt{x^2 - a^2}}{a}\right) - \frac{a^4 x}{16} \sqrt{x^2 - a^2}$
$- \frac{a^2 x}{8} (x^2 - a^2)^{\frac{3}{2}} + \frac{x^3}{6} (x^2 - a^2)^{\frac{3}{2}} + C.$

14. $y = \frac{x}{a^6 \sqrt{a^2 - x^2}} - \frac{2\sqrt{a^2 - x^2}}{a^6 x} - \frac{(a^2 - x^2)^{\frac{3}{2}}}{3a^6 x^3} + C.$

15. $y = -\frac{(a^2 - x^2)^{\frac{5}{2}}}{35a^4 x^7} (2x^2 + 5a^2) + C.$

16. $$y = \frac{5}{16a^7} \log \sqrt{\frac{a+x}{a-x}} + \frac{11x}{16a^6\,(a^2-x^2)} + \frac{3x^3}{8a^6\,(a^2-x^2)^2} + \frac{x^3}{6a^4\,(a^2-x^2)^3} + C.$$

17. $$y = -\frac{\sqrt{a^2-x^2}}{3a^4\,x^3}\,(a^2+2x^2) + C.$$

18. $$y = 2a^2 \log \frac{\sqrt{a^2-x^2}}{a} + \frac{2a^2\,x^2 - x^4}{2\,(a^2-x^2)} + C.$$

19. $$y = \frac{a^6}{6\,(a^2-x^2)^3} - \frac{3a^4}{4\,(a^2-x^2)^2} + \frac{3a^2}{2\,(a^2-x^2)} + \log \frac{\sqrt{a^2-x^2}}{a} + C.$$

20. $$y = \frac{3a^4}{8} \log \left(\frac{x+\sqrt{a^2+x^2}}{a}\right) + \frac{x\sqrt{a^2+x^2}}{8}\,(2x^2-3a^2) + C.$$

21. $$y = \frac{a^2}{a^2+x^2} + \log \frac{\sqrt{a^2+x^2}}{a} + \frac{1}{a} \tan^{-1}\frac{x}{a} + C.$$

22. $$y = -\frac{bx^2}{2a^2\,(a+bx^2)} + \frac{1}{2a^2} \log \frac{bx^2}{a+bx^2} + C.$$

EXAMPLES. XII. PAGE 62.

1. $$y = \frac{3a^2}{2} \sin^{-1}\left(\frac{x}{a} - 1\right) - \frac{\sqrt{2ax-x^2}}{2}\,(x+3a) + C.$$

2. $$y = \frac{5a^3}{2} \sin^{-1}\left(\frac{x}{a} - 1\right) - 4a^2\sqrt{2ax-x^2} - \frac{3}{2}a\,(x-a)\sqrt{2ax-x^2} + \frac{(2ax-x^2)^{\frac{3}{2}}}{3} + C.$$

3. $$y = a \sin^{-1}\left(\frac{x}{a} - 1\right) - \sqrt{2ax-x^2} + C.$$

4. $y = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a} - 1\right) + \frac{(x-a)\sqrt{2ax-x^2}}{2} + C.$

5. $y = \frac{a^3}{2}\sin^{-1}\left(\frac{x}{a} - 1\right) + \frac{a\,(x-a)\sqrt{2ax-x^2}}{2} - \frac{(2ax-x^2)^{\frac{3}{2}}}{3} + C.$

6. $y = \frac{x-2a}{a\sqrt{2ax-x^2}} + C.$

7. $y = \frac{a^2}{2}\ \log\left(\frac{x+a-\sqrt{2ax+x^2}}{a}\right) + \frac{(x+a)\sqrt{2ax+x^2}}{2} + C.$

8. $y = \frac{a}{2}\sin^{-1}\left(\frac{2x}{a} - 1\right) - \sqrt{ax-x^2} + C.$

EXAMPLES. XIII. PAGE 64.

1. $y = \log\,(2x+3+2\sqrt{2+3x+x^2}) + C.$

2. $y = \sin^{-1}\left(\frac{2x-1}{3}\right) + C.$

3. $y = \frac{9}{8}\sin^{-1}\left(\frac{2x-1}{3}\right) + \frac{2x-1}{4}\sqrt{2+x-x^2} + C.$

4. $y = -\frac{1}{2\,(a^2+x^3)^{\frac{2}{3}}} + C.$

5. $y = \frac{3}{2}(1+x^2)^{\frac{5}{3}}\left[\frac{(1+x^2)^2}{11} - \frac{1+x^2}{4} + \frac{1}{5}\right] + C.$

6. $y = \frac{(2x-3)\,(3x+2)}{3\,(x^2-3x+3)} + \frac{26}{3\sqrt{3}}\tan^{-1}\frac{2x-3}{\sqrt{3}} + C.$

7. $y = -\frac{bx^n}{na^2(a+bx^n)} + \frac{1}{na^2}\ \log\ \frac{bx^n}{a+bx^n} + C.$

8. $y = \frac{5}{132}\,(6x^4 - a^2x^2 - 5a^4)\,(a^2-x^2)^{\frac{1}{5}} + C.$

Mathematical Text Books

Published by

D. Van Nostrand Company

23 Murray Street
and
27 Warren Street

P. O. Box 1741 NEW YORK

Publishers' Announcement

¶ From the retail prices named in this list, a discount is made to schools, colleges and instructors connected therewith—a special introductory rate being furnished on application.

¶ The Publishers supply the instructor with a desk copy with all introductory orders.

Algebra and Trigonometry

Carnegie Technical Schools Series

8vo.	Half Leather.	250 Pages.	Price, $1.50 Net.
Algebra Section.	113 Pages.	Special Separate Edition.	$1.00 Net.

CONTENTS

ALGEBRA: Algebra and Arithmetic—Factoring—Equations—Graphics—The Binomial Theorem—Surds—Indeterminate—Equations—Quadratic — Equations—Logarithms—Inequalities — Progressions — Interest and Annuities. TRIGONOMETRY: The Relation between Angle and Line—Solution of Triangles—Solution of Oblique Triangles—Spherical Trigonometry—Application of Spherical Trigonometry.

PREFACE

Although this book has been designed to meet the specific needs of the Carnegie Technical Schools, the growing demand in the field of technical education for a form of mathematical instruction that will eliminate the purely speculative and concentrate the more utilitarian features of mathematical science, leads the author to believe that such a work as this will not be entirely inapt outside of the Carnegie Technical Schools.—S. S. K.

¶ (4) The development of the more elegant properties of circles, such as the angle of intersection, coaxial circles, similitude, etc.

¶ (5) The treatment of the conic sections. Use of the eccentric angle, etc. The equations of the conic sections are derived from a single definition.

¶ (6) The chapter on the elementary properties of confocal conics.

¶ (7) The proof of the formulas for the area of a triangle and of a polygon in terms of the co-ordinates of the verities.

¶ (8) The numerous original exercises. Among them are some of the most elegant properties of the conics. The diagrams have been made as suggestive as desirable in order to help the student in his work.

¶ It is believed that the author has succeeded in producing a work that is thoroughly modern, scientific, and up to date in every respect.

ELEMENTARY TREATISE ON THE

Differential and Integral Calculus

TWENTY-SECOND EDITION

12mo., Cloth. 451 Pages. 59 Figs. Price, $2.25

CONTENTS: First Principles—Differentiation of Algebraic and Transcendental Functions—Trigonometric Functions—Circular Functions—Limit—Derived Functions—Successive Differentials and Derivatives—Development of Functions—Evaluation of Indeterminate Forms—Functions of Two or More Variables—Change of the Independent Variable—Maxima and Minima of Functions of a Single Variable—Tangents, Normals and Asymptoles—Direction of Curvature—Singular Points—Tracing of Curves—Radius of Curvature, Evolutes and Involutes—Envelopes—Elementary Forms of Integration—Integration of Rational Fractions—Integration of Irrational Functions by Rationalization—Integration by Successive Reductions—Integration by Series—Successive Integration—Integration of Functions of Two Variables—Definite Integrals—Length of Curves—Areas of Plane Curves—Areas of Curved Surfaces—Volumes of Solids.

THE writer has adopted the method of *infinitesimals*, having learned from experience that the fundamental principles of the subject are made more intelligible to beginners by the method of infinitesimals than by that of *limits*, while in the practical applications of the Calculus the investigations are carried on entirely by the method of infinitesimals. At the same time, a thorough knowledge of the subject requires that the student should become acquainted with both methods; and for this reason, Chapter III is devoted exclusively to the method of limits. In this chapter all the fundamental rules for differentiating algebraic and transcendental functions are obtained by the method of limits, so that the student may compare the two methods.

Differential Equations

of the

First Species

By WILLIAM J. BERRY, C.E., M.S.
Instructor in Mathematics in the Polytechnic Institute of Brooklyn

12mo., Cloth. Illustrated. In Press.

THIS book is intended for engineers and students of engineering who desire to become proficient in the use of differential equations without spending too much time in study. To this end the equations are grouped according to certain standard forms, the solutions of which are carefully developed, and summarized in a plainly stated rule. Under each type, illustrative examples are given and fully worked out, no essential step being omitted. Numerous other examples are supplied to be solved by the student. While the proofs are mathematically rigid, the aim has been to make the presentation so simple that anyone with an elementary knowledge of the calculus can understand the reasoning and apply the rules. There is included a table of the forms with their rules, and a table of the more commonly used integrals. The chief points of the book are its Clearness, Conciseness, Convenience.

Descriptive Geometry

With Numerous Problems and Practical Applications

By Wm. S. HALL, C.E., E.M., M.S.
Professor of Graphics and Mining in Lafayette College, Easton, Pa.

With a Quarto Atlas of 18 Plates

8vo., Cloth. 76 Pages. Price, $3.50 Net.

CONTENTS

FIRST PRINCIPLES: Definitions—Projections or a Point—Revolutions of Vertical Plane—Projections of a Line—Representation of Planes—Notation and Conventions—Profile Planes. PROBLEMS ON THE POINT, LINE AND PLANE: Traces of a Line—Length of a Line—Points and Lines in Oblique Planes—Revolution of Points and Lines—Plane Angles—Planes—Additional Problems for Construction. CURVES AND TANGENTS TO CURVES: Generation and Classification of Lines—Curves and Tangents. SURFACES: Generation and Classification of Surfaces—Single Curved Surfaces and Tangent Planes—Surfaces of Revolution—Intersections and Developments—Warped Surfaces—Additional Problems on Surfaces.